PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Mean Flows Induced by Internal Gravity Wave Packets Propagating in a Shear Flow

R. Grimshaw

Phil. Trans. R. Soc. Lond. A 1979 **292**, 391-417
doi: 10.1098/rsta.1979.0067

MEAN FLOWS INDUCED BY INTERNAL GRAVITY WAVE PACKETS PROPAGATING IN A SHEAR FLOW

By R. GRIMSHAW
Department of Mathematics, University of Melbourne, Parkville, Victoria, 3052 *Australia*

(*Communicated by K. Stewartson, F.R.S. – Received* 5 *October* 1978)

CONTENTS

An inviscid, incompressible, stably stratified fluid occupies a horizontal channel, along which an internal gravity wave packet is propagating in the presence of a basic shear flow. By using a generalized Lagrangian mean formulation, the equation for wave action conservation is derived to describe the manner in which the basic flow affects the waves. Equations describing the second-order (in amplitude) wave-induced Lagrangian mean flows are obtained. Two kinds of applications are discussed: (i) steady mean flows, due to waves encountering an inhomogeneity in their environment, such as a varying channel depth; (ii) mean flows induced by modulations in the wave amplitude.

1. Introduction

Internal gravity waves are an important feature of the atmosphere and ocean. Although the linearized theory of these waves is now well understood and capable of describing many observational aspects, some of the most important consequences of internal gravity wave activity are due to nonlinear effects. In this paper we shall describe the manner in which a wave packet is affected by changes in its environment, and we shall calculate the mean flows induced by wave packets. The further question of how such induced mean flows affect the waves will be deferred to a later paper.

We shall consider internal gravity waves in an inviscid, incompressible and stably stratified fluid, bounded below by a rigid boundary, and above by a free surface. Our concern is thus with the oceanic environment in the first instance, although the cases when the upper surface is rigid, or removed to infinity, can easily be obtained from our analysis. The waves will be propagating relative to a basic state characterized by a density gradient and a horizontal shear flow. When the basic state is one of rest, the mean motions induced by modulated internal gravity waves in a channel have been discussed by Grimshaw (1977), Thorpe (1977) and Leonov *et al.* (1979), while some special cases have been considered by McIntyre (1973) and Chimonas (1978). One of our principal concerns is to evaluate the effect of a basic horizontal shear flow on these induced mean motions. Recently Thorpe (1978) has considered nonlinear effects for internal gravity waves in a shear flow, but he did not calculate the mean motions induced by modulated waves.

In §2 we introduce the equation of motion, and define the separation of scales and the averaging procedure which enables us to distinguish between the mean flow and the waves. The calculations are considerably simplified and clarified by using the generalized Lagrangian mean formulation recently proposed by Andrews & McIntyre (1978*a*). Although their formulation is exact we shall use it in an approximate sense, defined both by the smallness of the wave amplitude and by the separation of length scales between the mean flow and the waves.

It is well known that the effect of the environment on the waves is described by the equation for the conservation of wave action (Bretherton & Garrett 1969). The derivation of this equation for the present problem has been given by Hector *et al.* (1972). However, the use of the generalized Lagrangian mean formulation considerably simplifies the calculations (cf. Andrews & McIntyre, 1978*b*), which are described in §3.

The equations describing the induced mean flow are presented in §4. Instead of using the conventional method of finding an appropriate radiation stress tensor, we use the approach of Andrews & McIntyre (1978*a*) which calculates instead a quantity they have called the pseudomomentum, and obtains directly the equations for the Lagrangian mean flow in a fairly simple form. The corresponding Eulerian mean quantities can then be obtained by calculating the Stokes corrections.

Then in §§5 and 6 we describe some applications of our general theory. In §5 we consider the steady mean flows which arise when the basic flow is inhomogeneous in a single horizontal direction. Our principal application is to the propagation of internal gravity waves over varying bottom topography; it will be shown that whereas the Lagrangian mean velocities are zero in the absence of a basic shear flow, they do not vanish if any basic flow is present. Then in §6 we consider mean flows induced by modulated waves in the absence of any other inhomogeneities. Our principal conclusion here is that whenever the group velocity of the waves equals, or nearly equals, the basic shear flow, then there will be significantly large mean flows with a fine microstructure.

2. Generalized Lagrangian mean formulation

The Eulerian equations of motion for an inviscid, incompressible stably stratified fluid are

$$\frac{\partial u_i}{\partial x_i'} = 0, \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} = 0, \tag{2.1 a, b}$$

$$\rho \frac{\mathrm{d}u_i}{\mathrm{d}t} + \frac{1}{\beta}\frac{\partial p}{\partial x_i'} + \frac{\rho}{\beta}\delta_{i3} = 0, \tag{2.1 c}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i'}. \tag{2.1 d}$$

Here u_i are the velocity components, ρ is the density, p is the pressure, t is the time, and x_i' are Eulerian Cartesian coordinates; the roman subscript i takes the values 1, 2 or 3, and the summation convention is used; δ_{ij} is the Kroneker delta and the x_3'-axis is the vertical axis. The variables are non-dimensional, based on a length scale L (a typical wavelength), and a time scale N_1^{-1} where N_1 is a typical value of the Brunt–Väisälä frequency; the velocity scale is $N_1 L$, and the pressure scale $\rho_1 gL$, where ρ_1 is a typical value of the density. Then the parameter β is $N_1^2 L g^{-1}$, and is small in the Boussinesq approximation. It is convenient in the subsequent analysis to distinguish between horizontal coordinates $x_\alpha'(\alpha = 1, 2)$ and the vertical coordinate $z' = x_3'$ by employing greek indices for horizontal variables, while retaining roman indices for all three coordinates; similarly, u_α are the horizontal velocity components and $w = u_3$ is the vertical velocity. It will be assumed that the fluid occupies a horizontal channel, bounded below by a rigid boundary $z' = -h(x_\alpha')$, and above by the free surface $z' = \zeta(x_\alpha', t)$. The boundary conditions are

$$w + u_\alpha \,\partial h/\partial x_\alpha' = 0 \quad \text{on} \quad z' = -h(x_\alpha'), \tag{2.2 a}$$

$$\frac{\partial \zeta}{\partial t} + u_\alpha \frac{\partial \zeta}{\partial x_\alpha'} - w = 0 \quad \text{on} \quad z' = \zeta(x_\alpha', t), \tag{2.2 b}$$

$$p = 0 \quad \text{on} \quad z' = \zeta(x_\alpha', t). \tag{2.2 c}$$

To describe modulated waves we introduce a small parameter ϵ, and define the long horizontal variables and long time variable by

$$X_\alpha' = \epsilon x_\alpha', \quad T = \epsilon t. \tag{2.3}$$

Then if ϕ is any field variable (i.e. u_i, p, ρ, ζ), put

$$\phi(x_i', t) = \bar{\phi}(X_\alpha', T; z') + \phi'(X_\alpha', T; z' : \theta'), \tag{2.4 a}$$

where

$$\theta' = \epsilon^{-1}\Theta(X_\alpha', T), \tag{2.4 b}$$

and ϕ' is periodic in θ' with period 2π and zero mean. Also h is now assumed to be $h(X_\alpha')$. $\bar{\phi}$, the Eulerian mean, is an $O(1)$ quantity and varies on length and time scales of $O(\epsilon^{-1})$, while ϕ', the Eulerian perturbation, is an $O(a)$ quantity and is wavelike; here a is a small parameter which measures the wave amplitude. Both ϕ and ϕ' possess some modal structure in the vertical direction, and (2.4) describes a modulated wave on a background mean flow propagating in a horizontal waveguide. Note that

$$\bar{\phi}(x_i', t) = \langle \phi(x_i', t) \rangle = \frac{1}{2\pi}\int_0^{2\pi} \phi \,\mathrm{d}\theta', \tag{2.5}$$

defines an averaging operator, which represents a local average over the phase of the waves. It is conceptually useful to envisage θ' being replaced by $\theta'+\psi$, and with the averaging operator defined with respect to the ensemble label ψ, rather than the physical variable θ' (cf. Hayes (1970) or Andrews & McIntyre (1978*b*)). The general procedure now consists in substituting expressions such as (2.4*a*) into (2.1) and (2.2), and linearizing about the mean, to obtain equations governing the behaviour of the waves; the equations for the mean flow are obtained by applying the averaging operator to (2.1) and (2.2). This is essentially the method used by Hector *et al.* (1972), who derived the equation for conservation of wave action for internal gravity waves on a shear flow, and also the method used by Grimshaw (1977) who calculated the mean flows generated by internal gravity waves in the absence of a shear flow.

However, it transpires that the calculations are considerably simpler, both practically and conceptually, if we use the generalized Lagrangian mean flow (g.L.m.) formulation recently proposed by Andrews & McIntyre (1978*a*, *b*). Their theory is developed for a compressible fluid, and for a general class of averaging operators, although without detailed application to modal waves, but being an *exact* theory is readily adapted for use in the present context. We shall give a brief outline here, and refer the reader to Andrews & McIntyre (1978*a*, *b*) for further details, and for a discussion of the relation of the g.L.m. formulation to earlier theories. Let x_i be generalized Lagrangian coordinates and let $\xi_i(x_j, t)$ be the particle displacements defined so that

$$x_i' = x_i + \xi_i. \tag{2.6}$$

We then define a Lagrangian mean operator by

$$\overline{\phi}^{\mathrm{L}}(x_i, t) = \langle \phi(x_i+\xi_i, t)\rangle. \tag{2.7}$$

In physical terms, $\overline{\phi}$, the Eulerian mean, is the average over the phase of the waves taken at a fixed place, while $\overline{\phi}^{\mathrm{L}}$, the Lagrangian mean, is the average over the phase of the waves following the fluid motion. As shown by Andrews & McIntyre (1978*a*), this latter notion is made precise by requiring that

$$\langle \xi_i \rangle = 0, \tag{2.8}$$

whence it follows that x_i is a coordinate which moves with the Lagrangian mean velocity $\bar{u}_i^{\mathrm{L}}$ whenever the coordinate x_i' moves with the true velocity u_i. Since ξ_i is wavelike and $O(a)$, it follows from (2.4*a*, *b*) and (2.6) that we may put

$$\phi(x_i, t) = \overline{\phi}^{\mathrm{L}}(X_\alpha, T; z) + \hat{\phi}(X_\alpha, T; z, \theta), \tag{2.9a}$$

where

$$\theta = \epsilon^{-1}\Theta(X_\alpha, T), \tag{2.9b}$$

and $\hat{\phi}$ is periodic in θ with period 2π and zero mean. Note that θ is not identically equal to θ' (they differ by an $O(a)$ quantity), but Θ is unchanged, and so the averaging operator $\langle\ldots\rangle$ (2.5) may be regarded as averaging over the phase θ. Further by comparing (2.4*a*) with (2.9*a*), and expanding (2.4*a*) in ξ_i, it is readily shown that

$$\hat{\phi} = \phi' + \xi_i\,\partial\overline{\phi}/\partial x_i + O(a^2), \tag{2.10a}$$

$$\overline{\phi}^{\mathrm{L}} = \overline{\phi} + \overline{\phi}^{\mathrm{S}}, \tag{2.10b}$$

where

$$\overline{\phi}^{\mathrm{S}} = \left\langle \xi_i \frac{\partial \phi'}{\partial x_i}\right\rangle + \left\langle \tfrac{1}{2}\xi_i\,\xi_j \frac{\partial^2\overline{\phi}}{\partial x_i\,\partial x_j}\right\rangle + O(a^4). \tag{2.10c}$$

$\bar{\phi}^{S}$ is usually called the 'Stokes correction'. Finally, we note that the great usefulness of the g.L.m. formulation follows from the relation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\phi}{\partial t} + \bar{u}_i^{\mathrm{L}}\frac{\partial\phi}{\partial x_i}, \tag{2.11a}$$

and so

$$\left\langle\frac{\mathrm{d}\phi}{\mathrm{d}t}\right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle\phi\rangle. \tag{2.11b}$$

We shall not prove this result here, but refer the reader to Andrews & McIntyre (1978*a*). However, the result should be no surprise as $\mathrm{d}/\mathrm{d}t$ is the derivative following the fluid motion, and x_i has been defined by (2.8) to be the coordinate which moves with the velocity $\bar{u}_i^{\mathrm{L}}$.

The next step is to obtain the equations of motion in Lagrangian coordinates, x_i. First we introduce a mean density ρ^{L}, defined so that

$$\frac{\mathrm{d}\rho^{\mathrm{L}}}{\mathrm{d}t} + \rho^{\mathrm{L}}\frac{\partial\bar{u}_i^{\mathrm{L}}}{\partial x_i} = 0. \tag{2.12}$$

Next, (2.1*a*, *b*) together imply that

$$\rho J = \rho^{\mathrm{L}}, \tag{2.13a}$$

where

$$J = \det\{\partial x_i'/\partial x_j\}. \tag{2.13b}$$

But then (2.1*b*) implies that

$$\rho = \bar{\rho}^{\mathrm{L}} \quad \text{and} \quad \mathrm{d}\bar{\rho}^{\mathrm{L}}/\mathrm{d}t = 0. \tag{2.14}$$

Thus $\hat{\rho}$, the Lagrangian perturbation density, is identically zero, and from (2.13*a*), $\hat{J}$ is also identically zero. Substituting (2.6) into (2.13*b*) and expanding in ξ_i it follows that

$$\partial\xi_i/\partial x_i = O(a^2), \tag{2.15}$$

while the same procedure applied to (2.13*a*), and use of (2.15) shows that

$$\rho^{\mathrm{L}} = \bar{\rho}^{\mathrm{L}}\left\{1 - \left\langle\frac{1}{2}\frac{\partial^2}{\partial x_i\,\partial x_j}(\xi_i\,\xi_j)\right\rangle + O(a^4)\right\}. \tag{2.16}$$

Hence, by using (2.12) and (2.14), it follows that

$$\frac{\partial\bar{u}_i^{\mathrm{L}}}{\partial x_i} = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{1}{2}\frac{\partial^2}{\partial x_i\,\partial x_j}(\xi_i\,\xi_j)\right\rangle + O(a^4). \tag{2.17}$$

Also, from (2.10), and by using (2.15) and (2.16), it may be shown that

$$\beta\rho' = -\xi_i\,\partial\bar{\rho}/\partial x_i + O(a^2), \tag{2.18a}$$

and

$$\rho^{\mathrm{L}} = \bar{\rho} - \left\langle\frac{1}{2}\frac{\partial^2}{\partial x_i\,\partial x_j}(\bar{\rho}\xi_i\,\xi_j)\right\rangle + O(a^4). \tag{2.18b}$$

Here the Eulerian perturbation of the density is $\beta\rho'$ (rather than just ρ') as the density has been scaled by hydrostatic scales whereas the perturbation density should be scaled by dynamic factors. Similarly the Eulerian pressure perturbation is $\beta p'$ (rather than just p'). Now it follows from (2.6) that

$$u_i = \bar{u}_i^{\mathrm{L}} + \hat{u}_i, \tag{2.19a}$$

where

$$\hat{u}_i = \mathrm{d}\xi_i/\mathrm{d}t, \tag{2.19b}$$

and hence, by using (2.13*a*) and (2.14), the equation of motion (2.1*c*) becomes

$$\rho^{\mathrm{L}}\frac{\mathrm{d}\bar{u}_i^{\mathrm{L}}}{\mathrm{d}t} + \rho^{\mathrm{L}}\frac{\mathrm{d}^2\xi_i}{\mathrm{d}t^2} + \frac{J}{\beta}\frac{\partial p}{\partial x_i'} + \frac{\rho^{\mathrm{L}}}{\beta}\,\delta_{i3} = 0. \tag{2.20}$$

To make further progress, let K_{ij} be the i, jth co-factor of J, and so

$$K_{ij}\frac{\partial x'_i}{\partial x_k} = \delta_{jk}J = K_{ji}\frac{\partial x'_k}{\partial x_i}, \tag{2.21a}$$

or

$$K_{ij} = \delta_{ij}\left\{1+\frac{\partial \xi_k}{\partial x_k}\right\}-\frac{\partial \xi_j}{\partial x_i}+\tfrac{1}{2}\epsilon_{ilm}\epsilon_{jpq}\frac{\partial \xi_l}{\partial x_p}\frac{\partial \xi_m}{\partial x_q}. \tag{2.21b}$$

It may readily be shown that

$$K_{ij} = J\frac{\partial x_j}{\partial x'_i} \quad \text{and} \quad \frac{\partial K_{ij}}{\partial x_j} = 0. \tag{2.22}$$

Thus

$$J\frac{\partial p}{\partial x'_i} = \frac{\partial}{\partial x_j}(K_{ij}p), \tag{2.23}$$

and so, by extracting the mean and perturbed parts of (2.20), it follows that, by using (2.15),

$$\rho^{\mathrm{L}}\frac{\mathrm{d}\bar{u}_i^{\mathrm{L}}}{\mathrm{d}t}+\frac{1}{\beta}\frac{\partial}{\partial x_j}\langle K_{ij}p\rangle+\frac{\rho^{\mathrm{L}}}{\beta}\delta_{i3} = 0, \tag{2.24a}$$

and

$$\rho^{\mathrm{L}}\frac{\mathrm{d}^2\xi_i}{\mathrm{d}t^2}+\frac{1}{\beta}\frac{\partial}{\partial x_j}\left(-\bar{p}^{\mathrm{L}}\frac{\partial \xi_j}{\partial x_i}+\delta_{ij}\hat{p}\right) = O(a^2). \tag{2.24b}$$

The first of these equations, with (2.12) and (2.17), is the equation for the mean flow; the second equation, with (2.15), is the equation for the perturbed flow. By replacing $\hat{p}$ with $\beta p'$ by (2.10a), defining $\eta = \xi_3$ and using (2.13) and (2.14), (2.24b) becomes

$$\bar{\rho}^{\mathrm{L}}\frac{\mathrm{d}^2\xi_i}{\mathrm{d}t^2}+\frac{\partial p'}{\partial x_i}-\frac{\eta}{\beta}\frac{\partial \bar{p}^{\mathrm{L}}}{\partial x_i}-\xi_k\frac{\partial}{\partial x_i}\left(\bar{\rho}^{\mathrm{L}}\frac{\mathrm{d}\bar{u}_k^{\mathrm{L}}}{\mathrm{d}t}\right) = O(a^2). \tag{2.25}$$

In this form, the equation for the perturbed flow is readily identified with its Eulerian counterpart, noting that ρ' is defined by (2.18a). To this point, we have used the smallness of a, but have made no explicit use of the smallness of ϵ, and the equations are exact to all orders in ϵ.

It remains to consider the boundary conditions. We shall develop the theory suggested by Andrews & McIntyre (1978b). First if $F'(x'_i, t) = 0$ is a material boundary, then

$$\mathrm{d}F'/\mathrm{d}t = 0 \quad \text{when} \quad F' = 0. \tag{2.26}$$

If we write $F(x_i, t) \equiv F'_i(x_i+\xi_i, t)$, then this condition becomes

$$\mathrm{d}F/\mathrm{d}t = 0 \quad \text{when} \quad F = 0. \tag{2.27}$$

We apply this condition first to the rigid boundary $z' = -h(X'_\alpha)$, and set $F' \equiv z'+h(X'_\alpha)$. Then $F^{\mathrm{L}} = z+h(X_\alpha)+O(\epsilon^2a^2)$, and the mean, and perturbed parts of (2.27) imply that

$$\bar{w}^{\mathrm{L}}+\epsilon\bar{u}_\alpha^{\mathrm{L}}\,\partial h/\partial X_\alpha+O(\epsilon^2a^2) = 0 \quad \text{on} \quad z = -h(X_\alpha) \tag{2.28a}$$

and

$$\eta+\epsilon\xi_\alpha\,\partial h/\partial X_\alpha+O(\epsilon^2a^2) = 0 \quad \text{on} \quad z = -h(X_\alpha). \tag{2.28b}$$

Next, for the free boundary, (2.27) implies that $F = \bar{F}^{\mathrm{L}}$ and has no fluctuating part. Hence let $\bar{F}^{\mathrm{L}} = \bar{\zeta}^{\mathrm{L}}(X_\alpha, T)-z$, and then (2.27) implies that

$$\epsilon\frac{\partial \bar{\zeta}^{\mathrm{L}}}{\partial T}+\epsilon\bar{u}_\alpha^{\mathrm{L}}\frac{\partial \bar{\zeta}^{\mathrm{L}}}{\partial X_\alpha} = \bar{w}^{\mathrm{L}} \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(X_\alpha, T). \tag{2.29}$$

The pressure condition (2.2c) becomes

$$\bar{p}^{\mathrm{L}} = 0 \quad \text{and} \quad \hat{p} = 0 \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(X_\alpha, T). \tag{2.30}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

In Eulerian terms, the free surface is specified by $F' = \bar{\zeta}(X'_\alpha, T)+\zeta'-z'$, and since $F' = \bar{F}^{\mathrm{L}}$ it follows that

$$\zeta' = \eta-\epsilon\xi_\alpha\,\partial\bar{\zeta}/\partial X_\alpha+O(a^2) \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(X_\alpha, T) \tag{2.31a}$$

and

$$\bar{\zeta}^{\mathrm{L}} = \bar{\zeta}+\langle\xi_i\,\partial\zeta'/\partial x_i\rangle+O(\epsilon^2a^2) \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(X_\alpha, T). \tag{2.31b}$$

Also, by (2.10a),

$$\hat{p} = \beta p'+\xi_i\,\partial\bar{p}/\partial x_i+O(a^2), \tag{2.32}$$

and by combining (2.30), (2.31) and (2.32) we obtain the more familiar Eulerian boundary condition,

$$\beta p'+\zeta'\,\partial\bar{p}/\partial z = O(a^2) \quad \text{on} \quad z = \bar{\zeta}^{\mathrm{L}}(X_\alpha, T). \tag{2.33}$$

Thus the g.L.m. formulation has the advantage that the free surface boundary condition is just (2.30), and the free surface displacement ζ' may be calculated from (2.31) *a posteriori.*

3. Wave action conservation

The equations for the mean flow are (2.14), (2.17) and (2.24a). We put

$$\bar{\rho} = \rho_0(X_\alpha, T; z)+\beta a^2\bar{\rho}_2(X_\alpha, T; z)+O(a^4), \tag{3.1a}$$

$$\bar{p} = p_0(X_\alpha, T; z)+\beta a^2\bar{p}_2(X_\alpha, T; z)+O(a^4), \tag{3.1b}$$

$$\bar{u}_i^{\mathrm{L}} = u_{0i}(X_\alpha, T; z)+a^2\bar{u}_{2i}^{\mathrm{L}}(X_\alpha, T; z)+O(a^4), \tag{3.1c}$$

$$\bar{\zeta}^{\mathrm{L}} = \zeta_0(X_\alpha, T)+a^2\bar{\zeta}_2^{\mathrm{L}}(X_\alpha, T)+O(a^4). \tag{3.1d}$$

The factor β has been introduced into (3.1a, b) since the density and pressure have been scaled by hydrostatic considerations, but their Eulerian second order means should be scaled by dynamic factors. (The corresponding statement for the Lagrangian means is not true and $\bar{\rho}_2^{\mathrm{L}}$, for example, is $O(1)$ with respect to β.) Since the horizontal variable $X_\alpha = \epsilon x_\alpha$ is $O(\epsilon)$ compared with the vertical variable z, it is convenient to redefine $\bar{w}^{\mathrm{L}}$, etc., by $\epsilon\bar{w}^{\mathrm{L}}$, etc. The equations for the basic flow ρ_0, p_0, $u_{0\alpha}$, ϵw_0 are

$$\frac{\partial u_{0\alpha}}{\partial X_\alpha}+\frac{\partial w_0}{\partial z} = 0, \tag{3.2a}$$

$$\frac{\mathrm{D}\rho_0}{\mathrm{D}T} = 0, \tag{3.2b}$$

$$\rho_0\frac{\mathrm{D}u_{0\alpha}}{\mathrm{D}T}+\frac{1}{\beta}\frac{\partial p_0}{\partial X_\alpha} = 0, \tag{3.2c}$$

$$\epsilon^2\rho_0\frac{\mathrm{D}w_0}{\mathrm{D}T}+\frac{1}{\beta}\frac{\partial p_0}{\partial z}+\frac{1}{\beta}\rho_0 = 0, \tag{3.2d}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}T} = \frac{\partial}{\partial T}+u_{0\alpha}\frac{\partial}{\partial X_\alpha}+w_0\frac{\partial}{\partial z}. \tag{3.2e}$$

Since our scaling implies that the gradients of ρ_0 are $O(\beta)$ we put

$$\frac{\partial\rho_0}{\partial z} = -\beta\rho_0N^2, \quad \frac{\partial\rho_0}{\partial X_\alpha} = -\beta\rho_0M_\alpha, \tag{3.3}$$

where N is the Brunt–Väisälä frequency. The boundary conditions are derived from (2.28*a*), (2.29) and (2.30), and are

$$w_0 + u_{0\alpha}\,\partial h/\partial X_\alpha = 0 \quad \text{on} \quad z = -h(X_\alpha), \tag{3.4a}$$

$$\frac{\partial \zeta_0}{\partial T} + u_{0\alpha}\frac{\partial \zeta_0}{\partial X_\alpha} = w_0 \quad \text{on} \quad z = \zeta_0(X_\alpha, T), \tag{3.4b}$$

$$p_0 = 0 \quad \text{on} \quad z = \zeta_0(X_\alpha, T). \tag{3.4c}$$

The perturbed equations are (2.15) and (2.24*b*), or (2.25), where, to within an error of $O(a^2)$, we may replace ρ^{L} by ρ_0 etc. We now put (cf. (2.9*a*))

$$\xi_i = \xi_i(X_\alpha, T;\, z, \theta), \quad \text{etc.} \tag{3.4}$$

where θ is defined by (2.9*b*). Also we define the local frequency ω and the local wavenumber κ_α by

$$-\omega = \partial\Theta/\partial T, \; \kappa_\alpha = \partial\Theta/\partial X_\alpha, \tag{3.5a}$$

so that

$$\frac{\partial \kappa_\alpha}{\partial T} + \frac{\partial \omega}{\partial X_\alpha} = 0. \tag{3.5b}$$

Then, by using (3.2) and (3.3) the perturbed equations are

$$\kappa_\alpha \frac{\partial \xi_\alpha}{\partial \theta} + \frac{\partial \eta}{\partial z} = \epsilon I + O(a^2), \tag{3.6a}$$

$$\rho_0 \omega^{*2} \frac{\partial^2 \xi_\alpha}{\partial \theta^2} + \kappa_\alpha \frac{\partial p'}{\partial \theta} = \epsilon F_\alpha + O(a^2), \tag{3.6b}$$

$$\rho_0 \omega^{*2} \frac{\partial^2 \eta}{\partial \theta^2} + \frac{\partial p'}{\partial z} + \rho_0 N^2 \eta = \epsilon F_3 + O(a^2), \tag{3.6c}$$

where

$$\omega^* = \omega - \kappa_\alpha u_{0\alpha}. \tag{3.6d}$$

ω^* is the Doppler-shifted or intrinsic frequency. Here the $O(\epsilon)$ error terms are given by

$$I = -\,\partial \xi_\alpha/\partial X_\alpha, \tag{3.7a}$$

$$F_\alpha = 2\rho_0 \omega^* \frac{\mathrm{D}}{\mathrm{D}T}\left(\frac{\partial \xi_\alpha}{\partial \theta}\right) + \rho_0 \frac{\partial \xi_\alpha}{\partial \theta}\frac{\mathrm{D}\omega^*}{\mathrm{D}T} - \frac{\partial p'}{\partial X_\alpha} - \rho_0 M_\alpha \eta + O(\epsilon), \tag{3.7b}$$

$$F_3 = 2\rho_0 \omega^* \frac{\mathrm{D}}{\mathrm{D}T}\left(\frac{\partial \eta}{\partial \theta}\right) + \rho_0 \frac{\partial \eta}{\partial \theta}\frac{\mathrm{D}\omega^*}{\mathrm{D}T} + \xi_\alpha \frac{\partial}{\partial z}\left(\rho_0 \frac{\mathrm{D}u_{0\alpha}}{\mathrm{D}T}\right) + O(\epsilon). \tag{3.7c}$$

We now put

$$\eta = a\eta_1 \exp(\mathrm{i}\theta) + \text{c.c.} + O(a^2), \quad \text{etc.,} \tag{3.8}$$

where c.c. denotes complex conjugate. The equations (3.6) then become

$$\mathrm{i}\kappa_\alpha \xi_{1\alpha} + \frac{\partial \eta_1}{\partial z} = \epsilon I_1, \tag{3.9a}$$

$$-\rho_0 \omega^{*2} \xi_{1\alpha} + \mathrm{i}\kappa_\alpha p' = \epsilon F_{1\alpha}, \tag{3.9b}$$

$$\rho_0 (N^2 - \omega^{*2})\,\eta_1 + \frac{\partial p'}{\partial z} = \epsilon F_{13}, \tag{3.9c}$$

where I_1, $F_{1\alpha}$, F_{13} are obtained by substituting (3.8) into (3.7). Eliminating, we find that the equation for η_1, the vertical particle displacement, is

$$\frac{\partial}{\partial z}\left(\rho_0 \omega^{*2} \frac{\partial \eta_1}{\partial z}\right) + \rho_0 \kappa^2 (N^2 - \omega^{*2})\, \eta_1 = \epsilon M_1, \quad \kappa^2 = \kappa_\alpha \kappa_\alpha, \tag{3.10a}$$

where
$$M_1 = \kappa^2 F_{13} + \partial(\rho_0 \omega^{*2} I_1 + \mathrm{i}\kappa_\alpha F_{1\alpha})/\partial z. \tag{3.10b}$$

Also we have that
$$\xi_{1\alpha} = \frac{\mathrm{i}\kappa_\alpha}{\kappa^2}\frac{\partial \eta_1}{\partial z} + O(\epsilon), \tag{3.11a}$$

$$p_1' = \frac{\rho_0 \omega^{*2}}{\kappa^2}\frac{\partial \eta_1}{\partial z} + O(\epsilon). \tag{3.11b}$$

The relative simplicity of the equations (3.10) and (3.11) may be contrasted with the corresponding equations obtained from the Eulerian perturbation equations, which are usually formulated for the vertical velocity rather than the vertical particle displacement. The boundary conditions are (2.28*b*), (2.30), or

$$\eta_1 = -\epsilon \xi_{1\alpha}\, \partial h/\partial X_\alpha \quad \text{on} \quad z = -h(X_\alpha), \tag{3.12a}$$

$$\beta p_1' - \rho_0 \eta_1 = \epsilon \beta \rho_0 \xi_{1\alpha}\, \mathrm{D}u_{0\alpha}/\mathrm{D}T + O(\epsilon^2) \quad \text{on} \quad z = \zeta_0(X_\alpha, T). \tag{3.12b}$$

Here we have used (2.32) to express $\hat{p}$ in terms of p', and the basic flow equations (3.2*c*, *d*) to eliminate the gradients of p_0.

Ignoring the $O(\epsilon)$ correction term, (3.10*a*) is an ordinary differential equation in z alone, and with the boundary conditions (3.12) forms an eigenvalue problem for η_1 where ω (or the phase speed $c = \omega/\kappa$) is the eigenvalue (note that $\omega^* = \omega - \kappa_\alpha u_{0\alpha}$ (3.6*d*)), and κ_α is regarded as a fixed parameter (see also (6.2)). A sufficient condition that ω be real for real κ_α is that the Richardson number (the minimum of $N^2(\hat{\kappa}_\alpha\, \partial u_{0\alpha}/\partial z)^{-2}$, where $\hat{\kappa}_\alpha = \kappa_\alpha/\kappa$) be greater than $\frac{1}{4}$ everywhere, and we shall assume henceforth that this is the case. It may then be shown that the phase speed must be outside the range of $\hat{\kappa}_\alpha u_{0\alpha}$ (the component of the basic flow velocity in the direction of κ_α), or equivalently, that ω^* does not vanish for any z within the flow domain (Banks *et al.* 1976). The eigenvalue problem (3.10*a*) and (3.12) determines a dispersion relation $\omega = \omega(\kappa_\alpha; X_\alpha, T)$ where the dependence on X_α, T arises due to the parametric dependence of (3.10*a*) and (3.12) on X_α, T through $u_{0\alpha}$, etc. The dispersion relation combined with (3.5*b*) determines ω and κ_α as functions of X_α, T (or combined with (3.5*a*) it may be regarded as a partial differential equation for Θ). In general, there will be a number (possibly infinite) of such modes, and we shall fix our attention on one particular mode. Multiplying (3.10*a*) by η_1, and integrating across the channel, it follows that

$$\int_{-h}^{\zeta_0} \rho_0 \frac{\omega^{*2}}{\kappa^2}\left\{\left(\frac{\partial \eta_1}{\partial z}\right)^2 + \kappa^2 \eta_1^2\right\} \mathrm{d}z = \int_{-h}^{\zeta_0} \rho_0 N^2 \eta_1^2 \mathrm{d}z + \beta\left[\rho_0 \frac{\omega^{*4}}{\kappa^4}\left(\frac{\partial \eta_1}{\partial z}\right)^2\right]_{z=\zeta_0}. \tag{3.13}$$

The magnitude of the left hand side of this equation is one-quarter of the integrated energy density of the perturbed flow (the left hand side is one-half the kinetic energy and the right hand side is one-half the potential energy). Differentiating (3.13) with respect to κ_α it may be shown that

$$V_\alpha \mathscr{A} = \mathscr{B}_\alpha + U_{0\alpha}\mathscr{A}, \tag{3.14a}$$

where
$$\mathscr{A} = 2\int_{-h}^{\zeta_0} \rho_0 \frac{\omega^*}{\kappa^2}\left\{\left(\frac{\partial \eta_1}{\partial z}\right)^2 + \kappa^2\eta_1^2\right\} dz, \tag{3.14b}$$

$$\mathscr{B}_\alpha = 2\kappa_\alpha \int_{-h}^{\zeta_0} \rho_0 \frac{\omega^{*2}}{\kappa^4}\left(\frac{\partial \eta_1}{\partial z}\right)^2 dz, \tag{3.14c}$$

$$U_{0\alpha}\mathscr{A} = 2\int_{-h}^{\zeta_0} \frac{\rho_0 U_{0\alpha}\omega^*}{\kappa^2}\left\{\left(\frac{\partial \eta_1}{\partial z}\right)^2 + \kappa^2\eta_1^2\right\} dz, \tag{3.14d}$$

and
$$V_\alpha = \partial\omega/\partial\kappa_\alpha. \tag{3.14e}$$

Here V_α is the group velocity, and the magnitude of $\mathscr{A}$ is the integrated wave action density, since $|\mathscr{A}|$ is just the integral of one-half the local energy density divided by ω^*; also $|\mathscr{B}_\alpha|$ is just the integral of the local energy flux divided by ω^*, and $U_{0\alpha}|\mathscr{A}|$ represents the convection of wave action with the mean flow. These identities are due to Hector *et al.* (1972), who considered this eigenvalue problem from an Eulerian stand-point. It may easily be shown from (3.14*a*) that when ω^* is positive (i.e. the phase velocity c is greater than the component of the basic flow in the direction of the waves) then $V_\alpha\kappa_\alpha$ is less than ω (i.e. the component of the group velocity in the direction of the waves is less than the phase velocity), while when ω^* is negative, $V_\alpha\kappa_\alpha$ is greater than ω. Thus although c may not be in the range of the basic flow component in the wave direction, it is possible that the group velocity component may fall within this range. An example of this behaviour has recently been given by Thorpe (1978).

Turning now to the $O(\epsilon)$ terms in (3.10) and (3.12) we suppose that η_1 etc. are expanded in powers of ϵ; then at the first order in ϵ, M_1 in (3.10*a*) and the corresponding terms in (3.12) can be regarded as known. A necessary and sufficient condition that this inhomogeneous boundary value problem has a solution is the compatibility condition.

$$\int_{-h}^{\zeta_0}\left\{\eta_1\kappa^2F_{13} - \frac{\partial \eta_1}{\partial z}(\rho_0\omega^{*2}I_1 + i\kappa_\alpha F_{1\alpha})\right\}dz + \left[\rho_0\omega^{*2}\frac{\partial \eta_1}{\partial z}\xi_{1\alpha}\frac{\partial h}{\partial X_\alpha}\right]_{z=-h} - \beta\left[\rho_0\omega^{*2}\frac{\partial \eta_1}{\partial z}\xi_{1\alpha}\frac{Du_{0\alpha}}{DT}\right]_{z=\zeta_0} = 0. \tag{3.15}$$

This condition is derived by using the method of variation of parameters to solve (3.10*a*) and then applying the boundary conditions (3.12). If we now substitute (3.7) into (3.15) it may be shown by using (3.14), that the latter becomes

$$\frac{\partial \mathscr{A}}{\partial T} + \frac{\partial}{\partial X_\alpha}(V_\alpha\mathscr{A}) = 0. \tag{3.16}$$

This is the equation for conservation of wave action, and can be regarded as determining the complex amplitude of η_1, which is left undetermined by the leading order eigenvalue problem (see (6.1)). The result (3.16) was first derived for the present problem by Hector *et al.* (1972), who used an Eulerian standpoint (they considered only the case $w_0 = 0$). Although the calculations leading from (3.15) to (3.16) are extensive they are considerably simpler using the present Lagrangian formulation than in the Eulerian formulation.

The universality of the conservation of wave action in problems of this type is now well known (Bretherton & Garrett 1969). Recently Andrews & McIntyre (1978*b*) have developed an exact and general form for this conservation law. Their procedure is to form the scalar

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

product of $\partial\xi_i/\partial\theta$ with (2.20) and then apply the averaging operator. The outcome is the exact wave action equation

$$\epsilon\rho^{\mathrm{L}}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\frac{\partial\xi_i}{\partial\theta}\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right\rangle+\frac{\partial}{\partial x_j}\left\langle p\frac{\partial\xi_i}{\partial\theta}K_{ij}\right\rangle = 0. \tag{3.17}$$

As it stands this expression is of little value in the present context as the first three terms ($\mathrm{d}/\mathrm{d}t\langle\ldots\rangle$ and $j = 1, 2$) are $O(\epsilon)$, and so the remaining term ($j = 3$) requires the evaluation of η_1, p', etc., to $O(\epsilon)$. However if the expression is integrated over the channel we obtain

$$\epsilon\frac{\partial}{\partial T}\int_{-\bar{h}^{\mathrm{L}}}^{\bar{\zeta}^{\mathrm{L}}}\left\langle\rho^{\mathrm{L}}\frac{\partial\xi_i}{\partial\theta}\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right\rangle\mathrm{d}z+\epsilon\frac{\partial}{\partial X_\alpha}\int_{-\bar{h}}^{\bar{\zeta}^{\mathrm{L}}}\left\langle\rho^{\mathrm{L}}\bar{u}^{\mathrm{L}}_{\alpha}\frac{\partial\xi_i}{\partial\theta}\frac{\mathrm{d}\xi_i}{\mathrm{d}t}\right\rangle$$
$$+\epsilon\frac{\partial}{\partial X_\alpha}\int_{-\bar{h}}^{\bar{\zeta}^{\mathrm{L}}}\left\langle p\frac{\partial\xi_i}{\partial\theta}K_{i\alpha}\right\rangle\mathrm{d}z = 0. \tag{3.18}$$

Here $\bar{h}^{\mathrm{L}}$ is the Lagrangian mean of h (differs from $\bar{h}$ by $O(\epsilon^2a^2)$ terms), and we have used the boundary conditions (2.28a) and (2.29). We have also used the boundary condition

$$\frac{\partial\xi_i}{\partial\theta}K_{ij}\frac{\partial\bar{F}^{\mathrm{L}}}{\partial x_{ij}} = 0 \quad \text{on} \quad \bar{F}^{\mathrm{L}} = 0, \tag{3.19}$$

where $\bar{F}^{\mathrm{L}}$ is the Lagrangian mean of a material boundary $F = 0$ (on which $\mathrm{d}F/\mathrm{d}t = 0$ (2.27)), and is derived by differentiating the relation $F = \bar{F}^{\mathrm{L}}$ with respect to θ. The boundary condition (3.19) states that the component of the flux $\langle p\,\partial\xi_i/\partial\theta\,K_{ij}\rangle$ in the direction of the normal to the material boundary vanishes (see Andrews & McIntyre 1978b). Evaluating the integrands to leading order in ϵ and a, and with (3.14a), leads to (3.16) for $|\mathscr{A}|$. Of course, this derivation of the wave action equation, although elegant and elementary, has the disadvantage that it is assumed that the asymptotic expansions of ξ_i etc. in powers of ϵ exist. The previous derivation leading to (3.16) establishes that the wave action equation is a sufficient condition for the existence of this expansion, at least for the first two terms, and also contains an equation for the phase of $\mathscr{A}$.

Finally, in this section, we note that in the Boussinesq approximation $\beta\to 0$, by virtue of (3.3), ρ_0 becomes a constant in (3.10a) while the boundary condition (3.12b) becomes $\eta_1 = 0$ on $z = \zeta_0$. Our scaling has been designed to study internal waves. By contrast the surface wave has a frequency which scales with $\beta^{-\frac{1}{2}}$ and a pressure perturbation which scales with β^{-1}. If we rescale in this manner, and then take the limit $\beta\to 0$ our results reduce to the corresponding well known results for surface gravity waves (see Bretherton & Garrett 1969).

4. Mean flow evolution

The equations describing the evolution of the mean flow are (2.12), (2.17) and (2.24a). As shown by Andrews & McIntyre (1978a) the latter equation can be written in a number of alternative forms, by using the various identities (2.12a, b) and (2.22) for K_{ij}. In problems involving incompressible flow, it seems preferable to introduce the Eulerian mean pressure $\bar{p}$, and the Eulerian perturbation $\beta p'$. Then (2.24a) becomes

$$\rho^{\mathrm{L}}\frac{\mathrm{d}\bar{u}^{\mathrm{L}}_i}{\mathrm{d}t}+\frac{1}{\beta}\frac{\partial\bar{p}}{\partial x_i}+\frac{\rho^{\mathrm{L}}}{\beta}\delta_{i3}+\frac{\partial R_{ij}}{\partial x_j} = 0, \tag{4.1a}$$

where
$$R_{ij} = \left\langle\xi_j\frac{\partial p'}{\partial x_i}+\frac{1}{2}\frac{\xi_j\xi_k}{\beta}\frac{\partial^2\bar{p}}{\partial x_i\partial x_k}-\frac{1}{2}\frac{\partial}{\partial x_k}(\xi_j\xi_k)\frac{1}{\beta}\frac{\partial\bar{p}}{\partial x_i}\right\rangle+O(a^4). \tag{4.1b}$$

It is customary (Andrews & McIntyre 1978*a*) to call R_{ij} the radiation stress tensor, although in the present problem it is not the sole term describing the effect of the waves on the mean flow (there is another term in (2.17); cf. McIntyre (1973), who considered the special case of constant Brunt–Väisälä frequency N^2 in the Boussinesq approximation ($\beta \to 0$) in the absence of shear flow ($u_{0i} = 0$)). However, like (3.7), (4.1*a*) is not a very convenient equation to use in the present problem as for $i = 1, 2$ all the terms are $O(\epsilon)$ except those involving R_{i3}, and hence this latter term requires the evaluation of η_1, p_1', etc., to $O(\epsilon)$. Fortunately, Andrews & McIntyre (1978*a*) have presented an alternative mean flow equation which avoids this difficulty. This is obtained by first multiplying (2.20) by $\partial x_i'/\partial x_j$ and then applying the averaging operator $\langle\ldots\rangle$. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\bar{u}_i^{\mathrm{L}} + \frac{1}{\beta\bar{\rho}^{\mathrm{L}}}\frac{\partial \bar{p}^{\mathrm{L}}}{\partial x_i} + \frac{1}{\beta}\,\delta_{i3} = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathscr{P}_i + \mathscr{P}_j\frac{\partial \bar{u}_j^{\mathrm{L}}}{\partial x_i} + \frac{\partial}{\partial x_i}\langle\tfrac{1}{2}\hat{u}_j\,\hat{u}_j\rangle, \tag{4.2a}$$

where

$$\mathscr{P}_i = -\langle \hat{u}_j\,\partial\xi_j/\partial x_i\rangle. \tag{4.2b}$$

Being exact, (4.2*a*) is equivalent to (2.24*a*). We recall from (2.19*b*) that $\hat{u}_i = \mathrm{d}\xi_i/\mathrm{d}t$. Following Andrews & McIntyre (1978*a*, *b*), $\mathscr{P}_i$ is called the pseudomomentum. All the terms on the right hand side of (4.2*a*) now have the same order of magnitude as those on the left hand side with respect to the small parameter ϵ, and consequently the right hand side can be evaluated entirely from the leading order relations (3.11), where η_1 satisfies the leading order eigenvalue problem (2.10*a*) (see also (6.2)). The boundary conditions for the mean flow are (2.28*a*), (2.29) and (2.30) and contain no forcing terms due to the waves.

The outcome of evaluating $\mathscr{P}_i$, etc., is described below, where we have used (2.10*b*, *c*) to replace $\bar{p}^{\mathrm{L}}$ by $\bar{p}$. We find that, to leading order in a and ϵ,

$$\bar{\rho}^{\mathrm{L}}\frac{\mathrm{D_L}\bar{u}_\alpha^{\mathrm{L}}}{\mathrm{D}T} + \frac{1}{\beta}\frac{\partial \bar{p}}{\partial X_\alpha} = a^2\mathscr{R}_\alpha \tag{4.3a}$$

and

$$\frac{\mathrm{D_L}}{\mathrm{D}T} = \frac{\partial}{\partial T} + \bar{u}_\alpha^{\mathrm{L}}\frac{\partial}{\partial X_\alpha} + \bar{w}^{\mathrm{L}}\frac{\partial}{\partial z}, \tag{4.3b}$$

and the wave forcing term $\mathscr{R}_\alpha$ is given by

$$\mathscr{R}_\alpha = \frac{\mathrm{D}}{\mathrm{D}T}(\kappa_\alpha\mathscr{F}) + \mathscr{F}\kappa_\beta\frac{\partial u_{0\alpha}}{\partial X_\beta} + \frac{\partial}{\partial X_\alpha}\left\{\rho_0 N^2|\eta_1|^2 - \rho_0\omega^*|\eta_1|^2 - \rho_0\frac{\omega^{*2}}{\kappa^2}\left|\frac{\partial\eta_1}{\partial z}\right|^2\right\} + \tfrac{1}{2}\beta M_\alpha\omega^*\mathscr{F}, \tag{4.4a}$$

and

$$\mathscr{F} = 2\rho_0\omega^*\{|\eta_1|^2 + 1/\kappa^2|\partial\eta_1/\partial z|^2\}. \tag{4.4b}$$

Here $\mathscr{F}$ is the wave action density, and the integral of $\mathscr{F}$ across the channel is $\mathscr{A}$ (3.14*b*). For the vertical component of (4.2*a*) the simplest expression is obtained by replacing $\bar{p}^{\mathrm{L}}$ by $\bar{p}$, and $\bar{\rho}^{\mathrm{L}}$ by $\bar{\rho}$ (by using (2.16) and (2.18*b*)). The result is

$$\frac{1}{\beta}\frac{\partial\bar{p}}{\partial z} + \frac{\bar{\rho}}{\beta} = -a^2\frac{\partial}{\partial z}\{2\rho_0\omega^{*2}|\eta_1|^2\}, \tag{4.5}$$

which can also be obtained by applying the averaging operator $\langle\ldots\rangle$ directly to the vertical component of the Eulerian equations (2.1*c*). The remaining equations are (2.12), and (2.17) which simplifies to

$$\frac{\partial\bar{u}_\alpha^{\mathrm{L}}}{\partial X_\alpha} + \frac{\partial\bar{w}^{\mathrm{L}}}{\partial z} = a^2\frac{\mathrm{D}}{\mathrm{D}T}\left(\frac{\partial^2}{\partial z^2}|\eta_1|^2\right), \tag{4.6}$$

correct to leading order in a and ϵ. The relation between ρ^{L} and $\bar{\rho}$, or $\bar{\rho}^{\mathrm{L}}$ is (2.18b), or (2.16) which simplifies to

$$\rho^{\mathrm{L}} = \bar{\rho} - a^2 \frac{\partial^2}{\partial z^2}\{\rho_0 |\eta_1|^2\}, \tag{4.7a}$$

or

$$\rho^{\mathrm{L}} = \bar{\rho}^{\mathrm{L}} - a^2 \rho_0 \frac{\partial^2}{\partial z^2} |\eta_1|^2. \tag{4.7b}$$

Equations (2.14), (4.3a), (4.5)–(4.7) form the complete set of equations for the mean flow quantities $\bar{u}^{\mathrm{L}}_{\alpha}$, $\epsilon \bar{w}^{\mathrm{L}}$, $\bar{p}$ and $\bar{\rho}^{\mathrm{L}}$. The boundary conditions are (2.28a), (2.29) and (2.30); since the latter involves $\bar{p}^{\mathrm{L}}$ we must use (2.10a, b) to relate $\bar{p}$ and $\bar{p}^{\mathrm{L}}$. We find that

$$\bar{p}^{\mathrm{L}} = \bar{p} + \beta \bar{p}^{\mathrm{S}} \tag{4.8a}$$

and

$$\bar{p}^{\mathrm{S}} = a^2(\omega^* \mathscr{F} - \rho_0 N^2 |\eta_1|^2). \tag{4.8b}$$

To find the corresponding Eulerian mean velocity we must use (2.10b, c) to evaluate the Stokes velocity.

$$\bar{u}^{\mathrm{S}}_{\alpha} = a^2 \frac{\partial}{\partial z}\left\{\frac{\kappa_\alpha \omega^*}{\kappa^2} \frac{\partial}{\partial z} |\eta_1|^2 - 2|\eta_1|^2 \frac{\partial u_{0\alpha}}{\partial z}\right\} + a^2 |\eta_1|^2 \frac{\partial^2 u_{0\alpha}}{\partial z^2}, \tag{4.9a}$$

$$\bar{w}^{\mathrm{S}} = \frac{\mathrm{D}}{\mathrm{D}T}\left(\frac{\partial}{\partial z}|\eta_1|^2\right) + \frac{\partial}{\partial X_\alpha}\left(|\eta_1|^2 \frac{\partial u_{0\alpha}}{\partial z}\right) + \frac{\partial u_{0\alpha}}{\partial X_\alpha}\frac{\partial}{\partial z}|\eta_1|^2 - \frac{\partial}{\partial X_\alpha}\left(\frac{\kappa_\alpha \omega}{\kappa^2}\frac{\partial}{\partial z}|\eta_1|^2\right), \tag{4.9b}$$

and

$$\bar{u}_\alpha = \bar{u}^{\mathrm{L}}_{\alpha} - a^2 \bar{u}^{\mathrm{S}}_{\alpha}, \; \bar{w} = \bar{w}^{\mathrm{L}} - a^2 \bar{w}^{\mathrm{S}}. \tag{4.9c}$$

It is now readily established that

$$\frac{\partial \bar{u}^{\mathrm{S}}_{\alpha}}{\partial X_\alpha} + \frac{\partial \bar{w}^{\mathrm{S}}}{\partial z} = a^2 \frac{\mathrm{D}}{\mathrm{D}T}\left(\frac{\partial^2}{\partial z^2}|\eta_1|^2\right), \tag{4.10}$$

and comparing this with (4.6), we can confirm that the mean Eulerian velocity $\bar{u}_\alpha$, $\bar{w}$ is divergence free; of course, this result follows immediately from (2.1a). If (4.9c) is substituted into (4.3a) we obtain an equation for the mean Eulerian velocity $\bar{u}_\alpha$; it can be verified that the result is equivalent to directly averaging the Eulerian equation (2.1c). From (2.31a, b) we obtain the equations relating the mean Eulerian free surface displacement $\bar{\zeta}$ to the mean Lagrangian free surface displacement $\bar{\zeta}^{\mathrm{L}}$:

$$\bar{\zeta}^{\mathrm{S}} = 2\beta \left[\frac{\rho_0 \omega^{*2}}{\kappa^2}\left|\frac{\partial \eta_1}{\partial z}\right|^2\right]_{z=\zeta_0} \tag{4.11a}$$

and

$$\bar{\zeta} = \bar{\zeta}^{\mathrm{L}} - a^2 \bar{\zeta}^{\mathrm{S}}. \tag{4.11b}$$

If the mean flow equations (4.2a) are integrated across the channel, and we use the boundary conditions (2.28a), (2.29) and (2.30) (with (4.8a, b)), we find that

$$\frac{\partial}{\partial T}\left\{\int_{-h}^{\bar{\zeta}^{\mathrm{L}}} \rho^{\mathrm{L}} \bar{u}^{\mathrm{L}}_{\alpha}\,\mathrm{d}z\right\} + \frac{\partial}{\partial X_\beta}\left\{\int_{-h}^{\bar{\zeta}^{\mathrm{L}}} \rho^{\mathrm{L}} \bar{u}^{\mathrm{L}}_{\beta} \bar{u}^{\mathrm{L}}_{\alpha}\,\mathrm{d}z\right\} + \frac{\partial}{\partial X_\alpha}\left\{\int_{-h}^{\bar{\zeta}^{\mathrm{L}}} \frac{\bar{p}}{\beta}\,\mathrm{d}z\right\} - \frac{\partial h}{\partial X_\alpha}\left[\frac{1}{\beta}\bar{p}\right]_{z=-h}$$
$$= -a^2 \frac{\partial}{\partial X_\beta}(\mathscr{B}_\beta \kappa_\alpha) - a^2 \frac{\partial}{\partial X_\alpha}\left(\beta\left[\frac{\rho_0 \omega^{*4}}{\kappa^4}\left|\frac{\partial \eta_1}{\partial z}\right|^2\right]_{z=\zeta_0}\right). \tag{4.12}$$

Here $\mathscr{B}_\alpha$ is defined by (3.14c), and we have made use of (3.13), the equation obtained from (3.13) by differentiation with respect to X_α, and the wave action conservation equation (3.16). (4.12) can also be simply derived by integrating the alternative mean flow equation (4.1a, b) across the channel. From (3.14a), $\mathscr{B}_\alpha = (V_\alpha - U_{0\alpha})\mathscr{A}$, where V_α is the group velocity, and $U_{0\alpha}$, defined by (3.14*d*), can be regarded as an integrated form of the basic flow, weighted with the wave action density. The tensor $\mathscr{B}_\beta \kappa_\alpha$ is similar to radiation stress tensors encountered in the forcing of mean flows by waves in unbounded media (see, for example, Garrett (1968), Dewar (1970), Bretherton (1971) or Grimshaw (1978)). It is the dominant term in the right hand side of (4.12) for internal gravity waves in the Boussinesq approximation ($\beta \to 0$); however, for surface gravity waves, there is also a significant contribution from the second term on the right hand side of (4.12) (see Garrett (1968)). Also, integrating (4.6) across the channel, and using (4.11*a*, *b*) we find that

$$\frac{\partial \bar{\zeta}}{\partial T}+\frac{\partial}{\partial X_\alpha}\left\{\int_{-h}^{\bar{\zeta}} \bar{u}_\alpha^{\mathrm{L}}\,\mathrm{d}z\right\} = 0, \tag{4.13}$$

while integrating (2.12) leads to

$$\frac{\partial}{\partial T}\left\{\int_{-h}^{\bar{\zeta}^{\mathrm{L}}} \rho^{\mathrm{L}}\,\mathrm{d}z\right\}+\frac{\partial}{\partial X_\alpha}\left\{\int_{-h}^{\bar{\zeta}^{\mathrm{L}}} \rho^{\mathrm{L}}\bar{u}_\alpha^{\mathrm{L}}\,\mathrm{d}z\right\} = 0. \tag{4.14}$$

Although no wave forcing terms appear explicitly in (4.13) or (4.14) there is an implied wave forcing term due to the fact that $\bar{\zeta}$, the mean Eulerian free surface displacement, appears in (4.13) rather $\bar{\zeta}^{\mathrm{L}}$. For internal gravity waves in the Boussinesq approximation, this difference can be ignored, but it is significant for surface gravity waves. The integrated Stokes flow may be simply calculated from (4.9*a*) and is

$$\int_{-h}^{\zeta_0} \bar{u}_\alpha^{\mathrm{S}}\,\mathrm{d}z = a^2\int_{-h}^{\zeta_0} |\eta_1|^2\,\frac{\partial^2 u_{0\alpha}}{\mathrm{d}z^2}\,\mathrm{d}z+2a^2\beta\left[\left\{\frac{\kappa_\alpha\,\omega^{*3}}{\kappa^4}-\beta\,\frac{\partial u_{0\alpha}}{\partial z}\,\frac{\omega^{*4}}{\kappa^4}\right\}\left|\frac{\partial \eta_1}{\partial z}\right|^2\right]_{z=\zeta_0}. \tag{4.15}$$

For internal gravity waves in the Boussinesq approximation ($\beta \to 0$) the significant term is the first term which vanishes in the absence of a shear flow.

Finally, we substitute (3.1) into (2.12), (2.16), (4.3*a*), (4.5), (4.6) and (4.7) to obtain the equations for the second order quantities $\bar{u}_{2\alpha}^{\mathrm{L}}$, etc. We find that

$$\rho_0\left\{\frac{\mathrm{D}}{\mathrm{D}T}\,\bar{u}_{2\alpha}^{\mathrm{L}}+\bar{u}_{2\gamma}^{\mathrm{L}}\,\frac{\partial u_{0\alpha}}{\partial X_\gamma}+\bar{w}_2^{\mathrm{L}}\,\frac{\partial u_{0\alpha}}{\partial z}\right\}+\beta\bar{\rho}_2^{\mathrm{L}}\,\frac{\mathrm{D}u_{0\alpha}}{\mathrm{D}T}+\frac{\partial \bar{p}_2}{\partial X_\alpha} = \mathscr{R}_\alpha, \tag{4.16a}$$

$$\frac{\partial \bar{p}_2}{\partial z}+\bar{\rho}_2^{\mathrm{L}} = -\frac{\partial}{\partial z}\,\{2\rho_0\omega^{*2}|\eta_1|^2-2\rho_0 N^2|\eta_1|^2\}-|\eta_1|^2\,\frac{\partial}{\partial z}\,(\rho_0 N^2), \tag{4.16b}$$

$$\frac{\partial \bar{u}_{2\alpha}^{\mathrm{L}}}{\partial X_\alpha}+\frac{\partial \bar{w}_2^{\mathrm{L}}}{\partial z} = \frac{\mathrm{D}}{\mathrm{D}T}\left(\frac{\partial^2}{\partial z^2}\,|\eta_1|^2\right), \tag{4.16c}$$

$$\frac{\mathrm{D}}{\mathrm{D}T}\,\bar{\rho}_2^{\mathrm{L}} = \rho_0\,\bar{u}_{2\alpha}^{\mathrm{L}} M_\alpha+\rho_0 N^2\bar{w}_2^{\mathrm{L}}, \tag{4.16d}$$

while the boundary conditions (2.28), (2.29) and (2.30) become

$$\bar{w}_2^{\mathrm{L}}+\bar{u}_{2\alpha}^{\mathrm{L}}\,\partial h/\partial X_\alpha = 0 \quad \text{on} \quad z = -h(X_\alpha), \tag{4.17a}$$

$$\frac{\partial \bar{\zeta}_2^{\mathrm{L}}}{\partial T}+\bar{u}_{2\alpha}^{\mathrm{L}}\,\frac{\partial \zeta_0}{\partial X_\alpha}+\frac{\partial}{\partial X_\alpha}\,(u_{0\alpha}\,\bar{\zeta}_2^{\mathrm{L}})+\bar{\zeta}_2^{\mathrm{L}}\,\frac{\partial u_{0\alpha}}{\partial z}\,\frac{\partial \zeta_0}{\partial X_2} = \bar{w}_2^{\mathrm{L}} \quad \text{on} \quad z = \zeta_0(X_\alpha, T), \tag{4.17b}$$

and

$$\beta\bar{p}_2-\rho_0\,\bar{\zeta}_2^{\mathrm{L}} = \beta\left\{-\frac{2\rho_0\,\omega^{*2}}{\kappa^2}\left|\frac{\partial \eta_1}{\partial z}\right|^2-\beta^2(2\rho_0\,\omega^{*2}-\rho_0 N^2)\,\frac{\omega^{*4}}{\kappa^4}\left|\frac{\partial \eta_1}{\partial z}\right|^2\right\} \quad \text{on} \quad z = \zeta_0(X_\alpha, T). \tag{4.17c}$$

Here we find it convenient to use the mean Lagrangian density $\beta\bar{\rho}_2^L$, rather than its Eulerian counterpart. The two are related by (4.7 *a*, *b*),

$$\bar{\rho}_2 = \bar{\rho}_2^L - \frac{\partial}{\partial z}\{\rho_0 N^2|\eta_1|^2\} - \rho_0 N^2 \frac{\partial}{\partial z}|\eta_1|^2. \tag{4.18}$$

In the Boussinesq limit ($\beta \to 0$) the terms involving β explicitly in (4.10) and (4.17) may be omitted (so that (4.17 *c*) becomes $\bar{\zeta}_2^L = 0$ on $z = \zeta_0$) and ρ_0 regarded as a constant.

5. Applications: (i) steady mean flows

We put $X = X_1$ and suppose that the basic flow depends only on z and X. We put $u_0 = u_{01}$, $v_0 = v_{02}$, $\bar{u}_2^L = \bar{u}_{21}^L$, $\bar{v}_2^L = \bar{u}_{22}^L$, and $\kappa_1 = l$, $\kappa_2 = m$. Then assuming that the wave variables η_1, etc., likewise depend only on z and X, it follows from (3.5 *b*) that ω and m are constant, and the wavenumber l is determined from the dispersion relation $\omega = \omega(l, m; X)$. The dependence of the waves on X, and the consequent forcing of mean flows is due to the inhomogeneity of the basic flow (e.g. by the dependence of the depth h on X). The equation for conservation of wave action (3.16) reduces to

$$V_1 \mathscr{A} = \text{constant}, \tag{5.1}$$

where V_1 is the component of group velocity in the X-direction and is given by (3.14 *c*).

To make further progress we need to know η_1 from the eigenvalue problem (3.10 *a*) and (3.12 *a*, *b*) (with the right hand sides of these equations replaced by zero). In general, this information must be obtained numerically. However, an approximate method of solving (3.10 *a*) is to assume that the vertical scale of the waves is much shorter than the vertical scale of the basic flow (ρ_0, u_0, etc.) and to use a W.K.B. approximation. Then η_1 is given approximately by

$$\eta_1 \approx A(\rho_0 \omega^{*2} n)^{-\frac{1}{2}} \sin \phi, \tag{5.2 a}$$

where

$$\phi = \int_{-h}^{z} n \, \mathrm{d}z, \tag{5.2 b}$$

and

$$n = \kappa(N^2/\omega^{*2} - 1)^{\frac{1}{2}}, \tag{5.2 c}$$

and we have applied the bottom boundary condition (3.12 *a*). Here n may be identified as the vertical wavenumber, and is real for internal gravity waves, and the amplitude A is a function of X alone. Consistently with the W.K.B. approximation, we use the Boussinesq approximation $\beta \to 0$ so that the top boundary condition (3.12 *b*) reduces to $\eta_1 = 0$ on $z = \zeta_0$. This implies that

$$\phi_0 = \int_{-h}^{\zeta_0} n \, \mathrm{d}z = r\pi, \quad r = 1, 2, 3, \ldots. \tag{5.3}$$

The integer r identifies the various modes; (5.3) is an approximation to the dispersion relation. Evaluating $\mathscr{A}$ and V_1 from (3.14) we find that (5.1) becomes

$$A^2 \left\{\frac{lr\pi}{\kappa^4} + \int_{-h}^{\zeta_0} \frac{u_0 N^2}{n\omega^{*3}} \, \mathrm{d}z\right\} = \text{constant}. \tag{5.4}$$

Once the behaviour of l with respect to X has been determined from the dispersion relation, the quantity in brackets in (5.4) is known, and (5.4) then determines the variation of A with X. We shall illustrate by considering three special cases.

(a) Zero basic flow

If u_0, v_0, $w_0 = 0$, then $\rho_0 = \rho_0(z)$. Further, if N^2 is a constant and the Boussinesq approximation is made, then (5.2a) is exact, n (5.2c) is independent of z and the dispersion relation (5.3) becomes $nH = r\pi$, where $H = \zeta_0 + h$ is the total depth. This is the solution obtained by Keller & Mow (1969) to describe shoaling internal waves (see also Dore (1970) and McKee (1973)); from (5.2c) κ varies inversely with H, and so the horizontal wavelength ($2\pi\kappa^{-1}$) is proportional to the total depth H; Keller & Mow (1969) have shown that this result is in good agreement with some experiments by Wunsch (1969); also the W.K.B. approximations agree with the exact solutions obtained by Wunsch (1969) for waves on a constant slope in the short wavelength limit. For normal incidence ($m = 0$ and $\kappa = l$) the wave amplitude $|A|$ varies as $H^{-\frac{3}{2}}$, or the amplitude of η_1 (proportional to $|A|n^{-\frac{1}{2}}$) varies as H^{-1}; as pointed out by McKee (1973), the amplitude of the pressure p_1' (3.11b) remains constant as H varies.

(b) Top-intensified basic flow

A simple extension of case (a) is to suppose that the upper surface is rigid ($\zeta_0 = 0$), $u_0 = u_0(z), v_0 = w_0 = 0$ where $u_0 = 0$ for $z < -d$, where d is a constant such that $0 > d > h(X)$ for all X. This condition excludes the presence of a beach ($h = 0$), and this case describes the propagation of waves over a gentle step. Assuming that N^2 is a constant, the dispersion relation (5.3) becomes

$$\kappa\left\{H_d\left(\frac{N^2}{\omega^2}-1\right)^{\frac{1}{2}}+\int_{-d}^{0}\left(\frac{N^2}{\omega^{*2}}-1\right)^{\frac{1}{2}}\mathrm{d}z\right\} = r\pi, \quad r = 1, 2, 3, \ldots, \tag{5.5a}$$

$$H_d = h-d. \tag{5.5b}$$

The effect of the shear is through the second term on the left hand side of (5.5a); this term can be neglected if $\kappa u_{\mathrm{m}} d \ll \omega H(1-\omega^2/N^2)$, and then κ varies as H_d^{-1} (here u_{m} is the maximum value of $|u_0(z)|$). Likewise $|A|$ then varies as $H_d^{-\frac{3}{2}}$ for normal incidence. Explicit calculations with the profile $u_0 \propto (z+d)$ show that the effect of a positive shear flow (i.e. $u_0 \geqslant 0$) is to decrease κ below H_d^{-1} for low frequencies ($\omega \approx 0$), and to increase κ above H_d^{-1} for high frequencies ($\omega \approx N$); a negative shear flow (i.e. $u_0 \leqslant 0$) has the opposite effect. There is a corresponding change in the behaviour of A, determined by (5.4); the second term on the left hand side of (5.4) can be neglected if $\kappa^2 u_{\mathrm{m}} dN \ll (r\pi)\,\omega^2(1-\omega^2/N^2)^{\frac{1}{2}}$. Generally this second term is significant only at low frequencies and decreases $|A|$ for positive shears, while increasing $|A|$ for negative shears, relative to $H_d^{-\frac{3}{2}}$.

(c) Uniform basic flow over a step

If u_0 is not zero at $z = -h$, the effect of the slope is to distort the isopycnals of the basic flow and ρ_0 will depend on X as well as z. To obtain some information on this case consider the propagation of waves over a gentle step (figure 1) at normal incidence ($m = 0$) and suppose that u_0 is a constant when H is a constant (i.e. regions 1 and 2 in figure 1). Then u_0 will remain a function of X alone and we may put

$$u_0 H = \text{constant} = M_0, \quad H = \zeta_0 + h. \tag{5.6}$$

We shall also assume that $v_0 = 0$. From (3.2b) the density ρ_0 will remain constant on the streamlines of the basic flow. Assuming that N^2 is a constant in regions 1 and 2 it follows that

$$N^2 H = \text{constant} = N_0^2. \tag{5.7}$$

Again by using the Boussinesq approximation ($\beta \to 0$), the dispersion relation is (5.3), and since n is independent of z we have $nH = r\pi$, when n is given by (5.2*c*). But now, since N^2 and u_0 depend on H through (5.6) and (5.7) the dependence of κ on H is an involved one, and is given by

$$\kappa^2 H^2 \left\{ \frac{N_0^2 H}{(\omega H - \kappa M_0)^2} - 1 \right\} = r^2 \pi^2. \tag{5.8}$$

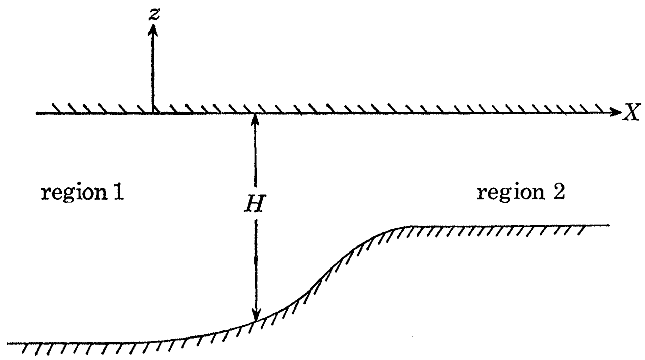

FIGURE 1. The coordinate system for case (*c*), uniform basic flow over a step.

The relation is graphed in figure 2*a*. The shaded region is forbidden, being the region where the quantity in brackets in (5.8) is negative. As the basic flow is reduced, $M_0 \to 0$, and the curves resemble those labelled (i) in figure 2*a*. When $\omega\kappa M_0 \to 0$ the basic flow is in the same direction as the waves and for small values of M_0, κ varies on the S-shaped curve (i), generally increasing as H decreases (this portion of the curve corresponds to the curve which would pertain if $M_0 = 0$), until a point is reached when the intrinsic frequency $\omega^*(\omega - \kappa u_0)$ becomes sufficiently small that a further decrease in H causes a decrease in κ. When $\omega\kappa M_0 < 0$ the basic flow opposes the waves and for small values of M_0, κ varies on the loop curve (i), either increasing or decreasing as H decreases, depending on the initial value of κ in region 1. For moderate or large values of M_0, the curves resemble those labelled (ii) in figure 2*a*. The loop curve ultimately disappears as M increases and the S-shaped curve is replaced by a monotonic curve on which κ decreases as H decreases. The variation of the amplitude with H is determined from (5.4); for normal incidence in the present case this simplifies to

$$A^2\{\omega H \kappa^{-3} + M_0 H^2 (r\pi)^{-2}/(\omega H - \kappa M_0)\} = \text{constant}. \tag{5.9}$$

This relationship is graphed in figure 2*b*, which shows the variation of $|A(\rho_0 \omega^{*2} n)^{-\frac{1}{2}}|$, the amplitude of η_1, with H (the curves have been arbitrarily normalized so that this amplitude is 1 when $H\omega^2/N^2$ is 0.5). Each curve shown corresponds to a curve in figure 2*a*, as designated. For $\omega\kappa M_0 > 0$ (the basic flow in the same direction as the waves) $|A(\rho_0 \omega^{*2} n)^{-\frac{1}{2}}|$ generally decreases as H decreases, although as $M_0 \to 0$, the variation of $|A(\rho_0 \omega^{*2} n)^{-\frac{1}{2}}|$ with H will contain an S-shape similar to the curve (i) for κ with H in figure 2*a*. For $\omega\kappa M_0 < 0$ (the basic flow opposes the waves) $|A(\rho_0 \omega^{*2} n)^{-\frac{1}{2}}|$ follows the crescent shaped curves in figure 2*b*. These show that $|A| \to \infty$ at the two values of H where $d\kappa/dH \to \infty$ in figure 2*a*; these two points are the values of H where the bracket in (5.4), or (5.9) vanishes, and corresponds to values of H where the group velocity vanishes. These are caustics where the waves are reflected. The analysis of this paper fails at a caustic, and an extended theory, such as that described by McKee (1974) for internal waves on a transverse current, is needed.

Turning next to the equations for the wave induced second order mean flow (4.16), we see that these become in the present context.

$$\rho_0 \frac{\partial}{\partial X}(u_0 \bar{u}_2^{\mathrm{L}}) + \rho_0 \bar{w}_2^{\mathrm{L}} \frac{\partial u_0}{\partial z} + \rho_0 w_0 \frac{\partial \bar{u}_2^{\mathrm{L}}}{\partial z} + \beta \bar{\rho}_2^{\mathrm{L}} \left(u_0 \frac{\partial u_0}{\partial X} + w_0 \frac{\partial u_0}{\partial z} \right) + \frac{\partial \bar{p}_2}{\partial X} = \mathscr{R}_1, \tag{5.10a}$$

$$\rho_0 \left(u_0 \frac{\partial \bar{v}_2^{\mathrm{L}}}{\partial X} + w_0 \frac{\partial \bar{v}_2^{\mathrm{L}}}{\partial z} + \bar{u}_0^{\mathrm{L}} \frac{\partial v_0}{\partial X} + \bar{w}_2^{\mathrm{L}} \frac{\partial v_0}{\partial z} \right) = \mathscr{R}_2, \tag{5.10b}$$

$$\frac{\partial \bar{p}_2}{\partial z} + \bar{\rho}_2^{\mathrm{L}} = -\frac{\partial}{\partial z} \{ 2\rho_0 \omega^{*2} |\eta_1|^2 - 2\rho_0 N^2 |\eta_1|^2 \} - |\eta_1|^2 \frac{\partial}{\partial z}(\rho_0 N^2), \tag{5.10c}$$

$$\frac{\partial \bar{u}_2^{\mathrm{L}}}{\partial X} + \frac{\partial \bar{w}_2^{\mathrm{L}}}{\partial z} = \left(u_0 \frac{\partial}{\partial X} + w_0 \frac{\partial}{\partial z} \right) \left(\frac{\partial^2}{\partial z^2} |\eta_1|^2 \right), \tag{5.10d}$$

$$\left(u_0 \frac{\partial}{\partial X} + w_0 \frac{\partial}{\partial z} \right) \bar{\rho}_2^{\mathrm{L}} = \rho_0 \bar{u}_2^{\mathrm{L}} M_1 + \rho_0 \bar{w}_2^{\mathrm{L}} N^2, \tag{5.10e}$$

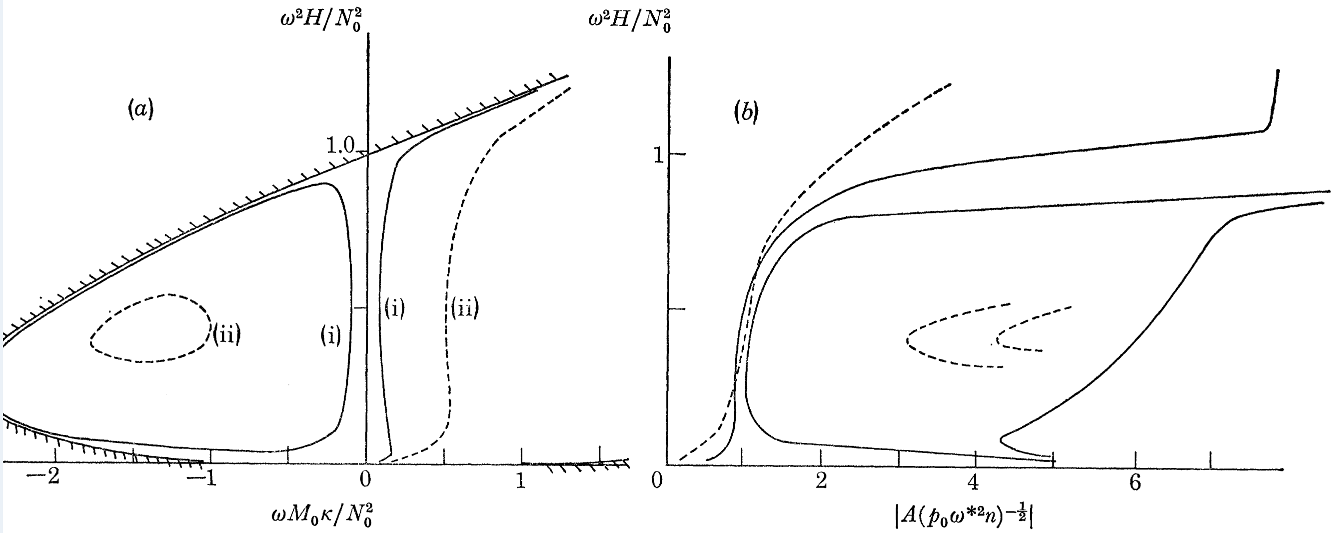

FIGURE 2. (*a*) A plot of $\omega^2 H N_0^{-2}$ against $\omega M_0 N_0^{-2} \kappa$ for two cases: (i) ——, $r^2\pi^2\omega^6 M_0^2 N_0^{-8} = 2.5 \times 10^{-5}$; (ii) -----, $r^2\pi^2\omega^6 M_0^2 N_0^{-8} = 1 \times 10^{-3}$. The hatched curve is where the quantity in brackets in (5.8) vanishes. (*b*) A plot of $\omega^2 H N_0^{-2}$ against $|A(\rho_0\omega^{*2}n)^{-\frac{1}{2}}|$ for two cases, (i) and (ii), as designated in (*a*). Both curves are normalized so that $|A(\rho_0\omega^{*2}n)^{-\frac{1}{2}}| = 1$ at $\omega^2 H N_0^{-2} = 0.5$, on the branch where $\omega M_0 \kappa > 0$.

while the boundary conditions are

$$\bar{w}_2^{\mathrm{L}} + \bar{u}_2^{\mathrm{L}} \frac{\partial h}{\partial X} = 0 \quad \text{on} \quad z = -h(X), \tag{5.11a}$$

$$\bar{u}_2^{\mathrm{L}} \frac{\partial \zeta_0}{\partial X} + \frac{\partial}{\partial X}(u_0 \bar{\zeta}_2^{\mathrm{L}}) + \bar{\zeta}_2^{\mathrm{L}} \frac{\partial u_0}{\partial z} \frac{\partial \zeta_0}{\partial X} = \bar{w}_2^{\mathrm{L}} \quad \text{on} \quad z = \zeta_0, \tag{5.11b}$$

and (4.17*c*). The equation for $\bar{v}_2^{\mathrm{L}}$ uncouples from the remaining equations and can be solved once $\bar{u}_2^{\mathrm{L}}$ and $\bar{w}_2^{\mathrm{L}}$ have been found. The forcing terms $\mathscr{R}_{1,2}$ are given by (4.4*a*) and simplify in the present context to

$$\mathscr{R}_1 = \frac{\partial}{\partial X}(l u_0 \mathscr{F}) + w_0 \frac{\partial}{\partial z}(l\mathscr{F}) + \tfrac{1}{2}\beta M_1 \omega^* \mathscr{F}$$
$$+ \frac{\partial}{\partial X} \left\{ \rho_0 N^2 |\eta_1|^2 - \rho_0 \omega^{*2} |\eta_1|^2 - \frac{\rho_0 \omega^{*2}}{\kappa^2} \left| \frac{\partial \eta_1}{\partial z} \right|^2 \right\}. \tag{5.12a}$$

$$\mathscr{R}_2 = \left(u_0 \frac{\partial}{\partial X} + \bar{w}_0 \frac{\partial}{\partial z} \right)(m\mathscr{F}) + l\mathscr{F} \frac{\partial v_0}{\partial X}. \tag{5.12b}$$

where $\mathscr{F}$ is defined by (4.4*b*).

(*d*) *Zero basic flow* (continued)

If $u_0, v_0, w_0 = 0$, then (5.10) and (5.11) have the exact solution

$$\bar{u}_2^L = 0, \quad \bar{w}_2^L = 0, \tag{5.13a}$$

$$\bar{p}_2 = \rho_0 N^2|\eta_1|^2 - \rho_0 \omega^2 |\eta_1|^2 - \rho_0 \frac{\omega^2}{\kappa^2}\left|\frac{\partial \eta_1}{\partial z}\right|^2, \tag{5.13b}$$

$$\bar{\rho}_2^L = \beta \rho_0 N^2 \omega \left\{|\eta_1|^2 + \frac{1}{\kappa^2}\left|\frac{\partial \eta_1}{\partial z}\right|^2\right\}, \tag{5.13c}$$

while (4.17*c*) determines $\bar{\zeta}_2^L$. The vanishing of the Lagrangian mean velocities $\bar{u}_2^L$, $\bar{w}_2^L$ is consistent with the presence of a beach (cf. Wunsch 1971). There is then a non-zero Eulerian mean velocity, which by (4.9*c*) is just the negative of the Stokes velocity (4.9*a*, *b*). In the present case the Stokes velocities are

$$\bar{u}^S = \frac{l\omega}{\kappa^2}\frac{\partial^2}{\partial z^2}|\eta_1|^2, \quad \bar{v}^S = \frac{m\omega}{\kappa^2}\frac{\partial^2}{\partial z^2}|\eta_1|^2, \tag{5.14a}$$

$$\bar{w}^S = -\frac{\partial}{\partial X}\left(\frac{l\omega}{\kappa^2}\frac{\partial}{\partial z}|\eta_1|^2\right). \tag{5.14b}$$

However, the transverse Lagrangian mean velocity $\bar{v}_2^L$ cannot be determined within the present framework; its determination requires the introduction of frictional considerations such as those used by Hogg (1971) in a study of the oblique incidence of internal waves onto a beach. For normal incidence it seems plausible that $\bar{v}_2^L$ is zero, but this may not be so for oblique incidence ($m \neq 0$); indeed Hogg (1971) has argued that the Eulerian mean transverse velocity $\bar{v}_2$ is then zero, and so $\bar{v}_2^L$ is not zero, but is given by $\bar{v}^S$ (5.14*a*). The Eulerian mean density is given by (4.18) and (5.13*c*); in the Boussinesq approximation ($\beta \to 0$), $\bar{\rho}_2^L \to 0$ but $\bar{\rho}$ remains non-zero. The mean Eulerian displacement of the isopycnal surface $z = z_0$ is $(\rho_0 N^2)^{-1}\bar{\rho}$ evaluated at $z = z_0$. Using the W.K.B. approximation (5.2) to evaluate (5.13) and (5.14), we find that $\bar{\rho}_2^L \approx 0$, and

$$\bar{p}_2 = -(n|A|^2/\kappa^2)\cos 2\phi, \quad \bar{\rho}_2 = (2N^2|A|^2/\omega^2)\sin 2\phi, \tag{5.15a}$$

$$\rho_0 \bar{u}^S = 2|A|^2(nl/\omega\kappa^2)\cos 2\phi, \; \rho_0 \bar{v}^S = (2|A|^2 ml/\omega\kappa^2)\cos 2\phi. \tag{5.15b}$$

Wunsch (1971) derived the mean flows generated by internal waves shoaling over a constant slope for constant Brunt–Väisälä frequency N; in the limit of small slope his solution reduces to (5.15). From (5.4), $|A|^2 l$ is proportional to κ^4, and for constant N^2, κ and n both vary as H^{-1}; hence $\bar{u}^S$ varies as H^{-3}, indicating substantial mean Eulerian flows as the beach is approached. Wunsch (1971) has discussed the implications of this in the oceanic context.

(*e*) *Top-intensified basic flow* (continued)

Here $w_0 = 0$, but $u_0(z)$ is not zero for $0 > z > -d$, where $d < h(X)$ for all X. It is now immediately apparent from (5.10*d*) that the Lagrangian mean velocities $\bar{u}_2^L$, $\bar{w}_2^L$ cannot be zero. To solve (5.10*d*) we put

$$\bar{u}_2^L = u_0\frac{\partial^2}{\partial z^2}|\eta_1|^2 + \frac{\partial(u_0\psi)}{\partial z}, \quad \bar{w}_2^L = -\frac{\partial(u_0\psi)}{\partial X}. \tag{5.16}$$

Then, on eliminating $\bar{p}_2$ and $\bar{\rho}_2^{\mathrm{L}}$ from the remaining equations, we find that

$$\frac{\partial}{\partial z}\left(\rho_0 u_0^2 \frac{\partial \psi}{\partial z}\right) + \rho_0 N^2 \psi = \frac{\partial}{\partial z}(\rho_0 \omega^{*2})\left\{|\eta_1|^2 + \frac{1}{\kappa^2}\left|\frac{\partial \eta_1}{\partial z}\right|^2\right\} + \frac{\partial}{\partial z}\left\{l u_0 \mathscr{F} - \rho_0 u_0^2 \frac{\partial^2}{\partial z^2}|\eta_1|^2\right\}, \tag{5.17a}$$

$$\psi = -\bar{\zeta}_2^{\mathrm{L}} \quad \text{at} \quad z = 0. \tag{5.17b}$$

Here $\bar{\zeta}_2^{\mathrm{L}}$ is given by (4.17*c*), and $\bar{\rho}_2^{\mathrm{L}} = -\rho_0 N^2 \psi$. In the region $z < -d$, $\bar{u}_2^{\mathrm{L}}$, $\bar{w}_2^{\mathrm{L}}$ are zero, but from (5.17*a*), it follows that $\bar{p}_2$ and $\bar{\rho}_2^{\mathrm{L}}$ are given by (5.13*b*, *c*), as in case (*a*). To make further progress in the region $z > -d$, we use the W.K.B. approximation (5.2*a*). Then (5.17*a*) simplifies to

$$\left\{\rho_0 u_0^2 \frac{\partial^2 \psi}{\partial z^2} + N^2 \psi\right\} \approx |A|^2 \left[\frac{2 l u_0}{\omega^*}\left(\frac{n^2}{\kappa^2} - 1\right) + \frac{4 n^2 u_0^2}{\omega^{*2}}\right] \sin 2\phi, \tag{5.18}$$

with the approximate solution,

$$\rho_0 \psi \approx \frac{|A|^2 u_0}{(N^2 - 4n^2 u_0^2)}\left[\frac{2l}{\omega^*}\left(\frac{n^2}{\kappa^2} - 1\right) + \frac{4n^2 u_0}{\omega^{*2}}\right] \sin 2\phi, \tag{5.19}$$

by using the boundary conditions (5.17*b*) with $\bar{\zeta}_2^{\mathrm{L}} = 0$ ($\bar{\zeta}_2^{\mathrm{L}}$ is $O(\beta)$ from (4.17*c*) and is thus zero in the W.K.B. approximation). For weak basic flows, ψ is $O(u_0)$ as $|u_0| \to 0$, and the principal contribution to the Lagrangian mean flow is the term $u_0\, \partial^2|\eta_1|^2/\partial z^2$ in $\bar{u}_2^{\mathrm{L}}$ (5.16). The approximate solution (5.19) fails whenever $N^2 = 4n^2 u_0^2$. If c_0 is the long wave phase speed (i.e. the limit of $c = \omega/\kappa$ as $\kappa \to 0$), then (5.2*c*) shows that $(c_0 - u_0)^2\, n^2 = N^2$; for a wave of mode number $2r$, this becomes $(c_0 - u_0)^2\, 4n^2 = N^2$. Hence the approximate solution (5.19) fails whenever the long wave phase speed c_0 vanishes for the long wave mode, with mode number $2r$. Indeed, if the left hand side of the exact equation (5.17*a*) is compared with the left hand side of (3.10*a*) (or (6.2*a*)) in the limit $\kappa \to 0$, then we see that the free solutions of (5.17*a*) (solutions when the right hand side vanishes) will be long waves if c_0 is zero for any long wave modes. Thus if c_0 vanishes for any long wave mode, there is no steady solution for the Lagrangian mean velocity, and we may expect instead a resonant growth of the mean velocities. Of course our hypotheses that the basic flow is stable will generally exclude this possibility as c_0, like c, must lie outside the range of u_0 which includes zero in the present case. However, the above discussion indicates the possibility of large Lagrangian mean flows whenever c_0 become small. The approximate solution (5.19) and also the exact solution of (5.17*a*) also fail if $c_0 = 2u_0$; this can only occur for isolated values of z, and is indicative of locally large mean velocities. The transverse Lagrangian mean velocity is readily determined from (5.10*b*) and (5.12*b*), and is

$$\bar{v}_2^{\mathrm{L}} = 2\rho_0 m \omega^* \left\{|\eta_1|^2 + \frac{1}{\kappa^2}\left|\frac{\partial \eta_1}{\partial z}\right|^2\right\} \quad \text{for} \quad z > -d, \tag{5.20}$$

but as in case (*a*), cannot be determined in $z < -d$ where u_0 is zero.

(*f*) *Uniform basic flow over a step* (continued)

Here $m = 0$, $v_0 = 0$, but $u_0 = u_0(X)$ and is given by (5.6). As in case (*b*), it is apparent from (5.10*d*) that $\bar{u}_2^{\mathrm{L}}$, $\bar{w}_2^{\mathrm{L}}$ cannot be zero. We can satisfy (5.10*b*) with $\bar{v}_2^{\mathrm{L}} = 0$, and (5.10*d*) by putting

$$\bar{u}_2^{\mathrm{L}} = u_0 \frac{\partial^2}{\partial z^2}|\eta_1|^2 + \frac{\partial}{\partial z}(u_0 \psi), \quad \bar{w}_2^{\mathrm{L}} = w_0 \frac{\partial^2}{\partial z^2}|\eta_1|^2 - \frac{\partial}{\partial X}(u_0 \psi). \tag{5.21}$$

Then (5.10*c*) can be satisfied by

$$\bar{\rho}_2^{\mathrm{L}} = -\rho_0 N^2 \psi. \tag{5.22}$$

Elimination of $\bar{p}_2$ in (5.10a, c) then yields an equation for ψ. The boundary conditions are (5.11) which simplify to

$$\psi = 0 \quad \text{on} \quad z = -h(X), \tag{5.23a}$$

$$\psi = \bar{\zeta}_2^{\mathrm{L}} \quad \text{on} \quad z = \zeta_0(X). \tag{5.23b}$$

Equations (5.21), (5.22) and (5.23) hold even when $u_0 = u_0(X, z)$ and not just for the present case. However, even in the present case the equation for ψ is quite complicated, and to make further progress we shall consider only regions 1 or 2 (figure 1) where u_0 is a constant and w_0 is zero. Under this condition the equation for ψ is again (5.17a). By using the W.K.B. approximation (5.2a), an approximate solution is again (5.19). The comments following equation (5.19) again apply. The variation of κ and $|A|$ with H are shown in figure 2; it can be shown from (5.19) and (5.21) that $\bar{u}_2^{\mathrm{L}}$ will be larger when H is smaller. For weak basic flows the first term in (5.21) dominates, and $\bar{u}_2^{\mathrm{L}}$ varies with X as $H^{-\frac{9}{2}}$.

6. Applications: (ii) mean flows induced by modulated waves

In this section we shall suppose that the basic flow is homogeneous (i.e. a function of z alone) and consider the mean flows induced by modulations in wave amplitude. Thus we let $u_{01} = u_0 = u_0(z)$, $u_{02} = v_0 = v_0(z)$, $w_0 = 0$, $\rho_0 = \rho_0(z)$ be a solution of the basic flow equations (3.2). We also choose h to be constant, and $\zeta_0 = 0$ so that the boundary conditions (3.4a, b) are satisfied. Consistently with these hypotheses we can satisfy (3.5b) by choosing κ_α and ω to be constants. There is now no explicit dependence on X_α, T in the eigenvalue problem for η_1 ((3.10a), (3.12a, b)) where the right hand sides are zero), and so we may write

$$\eta_1 = A(X_\alpha, T)f(z) + O(\epsilon), \tag{6.1}$$

where the modal function f satisfies the eigenvalue problem

$$\frac{\partial}{\partial z}\left(\rho_0\omega^{*2}\frac{\partial f}{\partial z}\right) + \rho_0\kappa^2(N^2-\omega^{*2})f = 0, \tag{6.2a}$$

$$f = 0 \quad \text{on} \quad z = -h, \tag{6.2b}$$

$$f - \beta\frac{\omega^{*2}}{\kappa^2}\frac{\partial f}{\partial z} = 0 \quad \text{on} \quad z = 0. \tag{6.2c}$$

We can assume that f is real-valued without any loss of generality. The dispersion relation is now just $\omega = \omega(\kappa_\alpha)$. We shall suppose that the waves travel in the positive X-direction ($X_1 = X$) and so put $\kappa_1 = \kappa$, $\kappa_2 = 0$. However, it follows from (3.14) that the group velocity is not necessarily in the X-direction, and will have a component in the Y-direction ($X_2 = Y$) whenever v_0 is not zero. We let $V_1 = V$ be the component in the X-direction, and $W = V_2$ be the component in the Y-direction; by using (6.1) in (3.14) it follows that both V, W are constants. Also, by using (6.1) in (3.14b) it follows that the wave action density $\mathscr{A}$ is equal to A^2 multiplied by a constant. Hence the equation for conservation of wave action (3.16) reduces to

$$\frac{\partial A}{\partial T} + V\frac{\partial A}{\partial X} + W\frac{\partial A}{\partial Y} = 0, \tag{6.3}$$

and so $A = A(X-VT, Y-WT)$; modulations in the wave amplitude propagate with the group velocity, and (6.1) describes a wave packet.

In the equations for the wave induced second order mean flow (4.16), the forcing terms are now functions of $(X-VT)$, $(Y-WT)$ and z. We shall seek solutions which likewise depend only in these variables, and so replace $\partial/\partial T$ by $-V\,\partial/\partial X - W\,\partial/\partial Y$. Again writing $\bar{u}^{\mathrm{L}}_{21} = \bar{u}^{\mathrm{L}}_{2}$, $\bar{u}^{\mathrm{L}}_{22} = \bar{v}^{\mathrm{L}}_{2}$, equations (4.16) reduce to

$$\rho_0\left\{(u_0-V)\frac{\partial \bar{u}^{\mathrm{L}}_2}{\partial X}+(v_0-W)\frac{\partial \bar{u}^{\mathrm{L}}_2}{\partial Y}+\bar{w}^{\mathrm{L}}_2\frac{\partial u_0}{\partial z}\right\}+\frac{\partial \bar{p}_2}{\partial X} = \mathscr{R}_1, \tag{6.4a}$$

$$\rho_0\left\{(u_0-V)\frac{\partial \bar{v}^{\mathrm{L}}_2}{\partial X}+(v_0-W)\frac{\partial \bar{v}^{\mathrm{L}}_2}{\partial Y}+\bar{w}^{\mathrm{L}}_2\frac{\partial v_0}{\partial z}\right\}+\frac{\partial \bar{p}_2}{\partial Y} = \mathscr{R}_2, \tag{6.4b}$$

$$\frac{\partial \bar{p}_2}{\partial z}+\bar{\rho}^{\mathrm{L}}_2 = |A|^2\left[-\frac{\partial}{\partial z}\{2\rho_0\omega^{*2}f^2-2\rho_0N^2f^2\}-f^2\frac{\partial}{\partial z}(\rho_0N^2)\right], \tag{6.4c}$$

$$\frac{\partial \bar{u}^{\mathrm{L}}_2}{\partial X}+\frac{\partial \bar{v}^{\mathrm{L}}_2}{\partial Y}+\frac{\partial \bar{w}^{\mathrm{L}}_2}{\partial z} = \frac{\partial^2 f^2}{\partial z^2}\left[(u_0-V)\frac{\partial}{\partial X}+(v_0-W)\frac{\partial}{\partial Y}\right]|A|^2, \tag{6.4d}$$

$$(u_0-V)\frac{\partial \bar{\rho}^{\mathrm{L}}_2}{\partial X}+(v_0-w)\frac{\partial \bar{\rho}^{\mathrm{L}}_2}{\partial Y} = \rho_0N^2\bar{w}^{\mathrm{L}}_2, \tag{6.4e}$$

while the boundary conditions are

$$\bar{w}^{\mathrm{L}}_2 = 0 \quad \text{on} \quad z=-h, \tag{6.5a}$$

$$(u_0-V)\frac{\partial \bar{\zeta}^{\mathrm{L}}_2}{\partial X}+(v_0-W)\frac{\partial \bar{\zeta}^{\mathrm{L}}_2}{\partial Y} = \bar{w}^{\mathrm{L}}_2 \quad \text{on} \quad z=0, \tag{6.5b}$$

and (4.17*c*). The forcing terms $\mathscr{R}_{1,2}$ are given by (4.4*a*) and simplify in the present context to

$$\mathscr{R}_1 = 2\rho_0\omega^*\kappa\left\{f^2+\frac{1}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right\}\left[(u_0-V)\frac{\partial}{\partial X}+(v_0-W)\frac{\partial}{\partial Y}\right]|A|^2$$
$$+\left\{\rho_0N^2f^2-\rho_0\omega^{*2}f^2-\frac{\rho_0\omega^{*2}}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right\}\frac{\partial}{\partial X}|A|^2, \tag{6.6a}$$

and
$$\mathscr{R}_2 = \left\{\rho_0N^2f^2-\rho_0\omega^{*2}f^2-\frac{\rho_0\omega^{*2}}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right\}\frac{\partial}{\partial Y}|A|^2. \tag{6.6b}$$

The Stokes velocities are given by (4.9*a*, *b*) while the Eulerian mean density is given by (4.18). Using these equations and (4.9*c*) we may reformulate the equations (6.4) in terms of Eulerian mean quantities. When u_0, v_0 are zero, the resulting equations then agree with those obtained by Grimshaw (1977), Thorpe (1977) and Leonov *et al.* (1978).

(*a*) *X-dependent modulations*

Suppose first that the amplitude contains no dependence on Y, and $A = A(X-VT)$. Then in (6.4), (6.5) and (6.6) we may put terms involving $\partial/\partial Y$ equal to zero. Equations (6.4*d*, *e*) can then be solved by putting $\bar{v}^{\mathrm{L}}_2 = 0$, and

$$\bar{u}^{\mathrm{L}}_2 = (u_0-V)|A|^2\frac{\partial^2 f^2}{\partial z^2}+|A|^2\frac{\partial}{\partial z}(u_0-V)\psi, \quad \bar{w}^{\mathrm{L}}_2 = -(u_0-V)\psi\frac{\partial}{\partial X}|A|^2, \tag{6.7a}$$

and
$$\bar{\rho}^{\mathrm{L}}_2 = -\rho_0N^2\psi|A|^2. \tag{6.7b}$$

Substitution into (6.4*a*, *c*) and elimination of $\bar{p}_2$ leads to the equation for ψ:

$$\frac{\partial}{\partial z}\left\{\rho_0(u_0-V)^2\frac{\partial\psi}{\partial z}\right\}+\rho_0N^2\psi = \mathscr{M}_1, \tag{6.8a}$$

where

$$\mathcal{M}_1 = \frac{\partial}{\partial z}(\rho_0\omega^{*2})\left\{f^2+\frac{1}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right\}$$
$$+\frac{\partial}{\partial z}\left\{2\rho_0\omega^*\kappa(u_0-V)\left(f^2+\frac{1}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right)-\rho_0(u_0-V)^2\frac{\partial^2 f^2}{\partial z^2}\right\}. \quad (6.8\,b)$$

The pressure is given by

$$\bar{p}_2+\rho_0(u_0-V)^2\frac{\partial\psi}{\partial z}|A|^2 = \left\{\rho_0 N^2 f^2-\rho_0\omega^{*2}f^2-\frac{\rho_0\omega^{*2}}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right\}|A|^2$$
$$+\left\{2\rho_0\omega^*\kappa(u_0-V)\left(f^2+\frac{1}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right)-\rho_0(u_0-V)^2\frac{\partial^2 f^2}{\partial z^2}\right\}|A|^2. \quad (6.9)$$

The boundary conditions (6.5) become

$$\psi = 0 \quad \text{at} \quad z = -h, \quad (6.10\,a)$$

$$\psi-\beta(u_0-V)^2\frac{\partial\psi}{\partial z} = \beta\left\{-[\omega^{*2}+2\omega^*\kappa(u_0-V)]\left[f^2+\frac{1}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right]+(u_0-V)^2\frac{\partial^2 f^2}{\partial z^2}\right\} \quad \text{at} \quad z = 0, \quad (6.10\,b)$$

while
$$\zeta_2^L = -\psi|A|^2 \quad \text{at} \quad z = 0. \quad (6.10\,c)$$

If the homogeneous equations for ψ (i.e. (6.8a) and (6.10a, b) with zero on the right hand sides) are compared with (6.2) in the limit $\kappa \to 0$, we see that the free solutions for ψ will be long waves if $V = c_0$, where c_0 is the phase speed of any long wave mode. When this occurs the inhomogeneous equations (6.8a) and (6.10a, b) cannot be solved for ψ, and instead there is a resonance between the wave packet and a long wave mode. Equations describing this resonance have been developed by Grimshaw (1977) for the case $u_0 = v_0 = 0$; the corresponding equations in the present case will be described elsewhere.

Equation (6.8a) has a singularity at any level of $z = z_s$ where $u_0(z_s) = V$. Thorpe (1978) has recently shown, for one particular case, that this phenomenon can occur, even though the phase speed c must lie outside the range of $u_0(z)$. In this case the homogeneous form of (6.8a) has the two solutions

$$\psi_\pm = (z-z_s)^{-\frac{1}{2}\pm i\mu}\{1+O(z-z_s)\}, \quad (6.11\,a)$$

where
$$\mu^2+\tfrac{1}{4} = N^2(\partial u_0/\partial z)^{-2}|_{z=z_s}, \quad (6.11\,b)$$

and we are assuming that ρ_0, u_0 are analytic functions of z near z_s; μ is real by our hypothesis that the basic flow is stable. The general solution of (6.8a) is then

$$\psi = A_+\psi_+ + A_-\psi_- + \psi_p, \quad (6.12\,a)$$

where
$$\psi_p = \psi_+\int_{z_s}^{z}\frac{\mathcal{M}_1\psi_-}{W}\,dz-\psi_-\int_{z_s}^{z}\frac{\mathcal{M}_1\psi_+}{W}\,dz, \quad (6.12\,b)$$

and
$$W = \rho_0(\partial u_0/\partial z)^2|_{z=z_s}. \quad (6.12\,c)$$

Now the modal function f is analytic at z_s, and so $\mathcal{M}_1$ (6.8b) is also analytic at $z = z_s$. It can then be shown from (6.11a) and (6.12b) that the particular solution ψ_p is analytic at z_s. Assuming that there is no long wave resonance, the imposition of the boundary conditions (6.10a, b) then determine $A\pm$ uniquely, and in general will be non-zero; they can both vanish only in the unlikely event that ψ_p satisfies both boundary conditions. Thus our solution for ψ will contain a singularity at $z = z_s$ described by (6.11*a*). From (6.7) $\bar{u}_2^{\mathrm{L}}$ and $\bar{\rho}_2^{\mathrm{L}}$ are proportional to $|z-z_s|^{-\frac{1}{2}\pm i\mu}$ near this singularity. Of course, our assumptions that lead to (6.8*a*) have failed for z near z_s. Nevertheless we can conclude that there will be substantial mean flows generated at such levels. The principal assumption which has failed is that $\partial/\partial T$ can be replaced by $-V\,\partial/\partial X$. Retaining the time derivative would lead to an analysis similar to that given by Booker & Bretherton (1967) for the time development of a critical level. On this basis we can conjecture that (6.12*a*) will hold outside a region in which $|z-z_s|$ is $O(T^{-1})$, while around the critical level, ψ will remain unsteady and become large.

Another aspect to the solution (6.12*a*) is its highly oscillatory character near $z = z_s$, where the vertical length scale is $\mu|z-z_s|^{-1}$. Further, it is apparent from (6.8*a*) that whenever $|u_0 - V|$ is small, ψ will have a vertical length scale of $O(|u_0-V|N^{-1})$. When there is no basic shear flow ($u_0 = 0$, $v_0 = 0$), Chimonas (1978) has obtained solutions for mean flows generated by internal gravity wave packets in a model of the planetary boundary layer, and has suggested that shear flow instabilities may be associated with the small vertical scale of the induced mean flow. Leonov *et al.* (1978) have commented that internal gravity wave packets in the ocean are responsible for the development of a vertical microstructure in the mean velocity and density fields, and have made some calculations for models of the ocean when there is no basic shear flow. In both these cases the vertical scale of the forced mean flows is $O(VN^{-1})$. The presence of a shear flow alters this scale to $O(|u_0-V|N^{-1})$; the microstructure is then localized to levels where $|u_0-V|$ is small, but the potentiality of realizing microstructure is increased as it may be easier to find circumstances when $|u_0-V|$ is small, rather than just V is small.

If $|u_0-V|$ is not small over the flow domain then we may use the W.K.B. approximation (5.2) to evaluate $\mathcal{M}_1$, and so obtain an approximate solution to (6.8*a*). The result is

$$\rho_0\psi \approx \frac{(u_0-V)}{[N^2-4n^2(u_0-V)^2]}\left[\frac{2\kappa}{\omega^*}\left(\frac{n^2}{\kappa^2}-1\right)+\frac{4n^2(u_0-V)}{\omega^{*2}}\right]\sin 2\phi. \tag{6.13}$$

This solution is ψ_p, and so the first term of the W.K.B. approximation satisfies the boundary conditions. The solution fails whenever the denominator vanishes, which can only occur at long wave resonance, $V = c_0$. However, the W.K.B. approximation involves the neglect of terms involving derivatives of u_0 etc. and it is precisely these terms which case ψ_p to not satisfy the boundary conditions. Including higher order terms in the W.K.B. approximations would lead to a term proportional to cos 2ϕ in ψ_p, and hence to the inclusion of $\psi_\pm$ in the solution for ψ. Here $\psi_\pm$ are given by (6.11*a*) when $u_0 - V$ vanishes within the flow domain; alternatively a W.K.B. approximation may be used to show that

$$\psi_\pm \approx (\rho_0(u_0-V)\,N)^{-\frac{1}{2}}\exp(\mathrm{i}\phi_\pm), \tag{6.14a}$$

where

$$\phi_\pm = \pm\int_{-h}^{z} N(u_0-V)^{-1}\,\mathrm{d}z. \tag{6.14b}$$

Except for a constant factor (6.11*a*) and (6.14*a*) agree near $z = z_s$. By way of illustration, a solution of (6.8*a*) has been obtained numerically for the case when N is a constant, the Boussinesq approximation is made, and u_0 is the linear show flow $U_1 zh^{-1}$, where U_1 is a constant. In the absence of a shear flow, (6.13) is exact; a typical solution when u_0 is not zero is shown in figure 3. (For the case shown, the solution when $U_1 = 0$ is larger due to the proximity of a long wave resonance).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

(*b*) *Transverse modulations*

We now allow the amplitude to depend on Y as well as X. For simplicity of presentation, we shall also assume that $v_0 = 0$, and so $W = 0$. Now put

$$\bar{u}_2^L = (u_0 - V)\,|A|^2\,\frac{\partial^2 f^2}{\partial z^2} + |A|^2\,\frac{\partial}{\partial z}\,(u_0 - V)\,\psi + \hat{u}, \tag{6.15a}$$

$$\bar{w}_2^L = -(u_0 - V)\,\psi\,\frac{\partial}{\partial X}\,|A|^2 + (u_0 - V)\,\frac{\partial \hat{w}}{\partial X}, \tag{6.15b}$$

$$\bar{\rho}_2^L = -\rho_0 N^2 \psi |A|^2 + \rho_0 N^2 \hat{w}. \tag{6.15c}$$

substitute into (6.4), and eliminate the variables in favour of $\hat{w}$. The result is

$$\frac{\partial^2}{\partial X^2}\left[\frac{\partial}{\partial z}\left(\rho_0 (u_0 - V)^2\,\frac{\partial \hat{w}}{\partial z}\right)\right] + \rho_0 N^2 \left[\frac{\partial^2 \hat{w}}{\partial X^2} + \frac{\partial^2 \hat{w}}{\partial Y^2}\right] = \mathscr{M}_2\,\frac{\partial^2 |A|^2}{\partial Y^2}, \tag{6.16a}$$

where
$$\mathscr{M}_2 = \rho_0 N^2 \psi - \frac{\partial}{\partial z}\,(\rho_0 \omega^{*2})\left\{f^2 + \frac{1}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right\}, \tag{6.16b}$$

and
$$\frac{\partial}{\partial z}\,(\rho_0 (u_0 - V)\,\hat{u}) = \rho_0 N^2 \hat{w} - \frac{\partial}{\partial z}\left\{\rho_0 (u_0 - V)\,\frac{\partial u_0}{\partial z}\,\hat{w}\right\}. \tag{6.16c}$$

The boundary conditions are

$$\hat{w} = 0 \quad \text{on} \quad z = -h, \tag{6.17a}$$

$$\hat{w} + \beta(u_0 - V)\,\{\hat{u} + \partial u_0/\partial z\,\hat{w}\} = 0 \quad \text{on} \quad z = 0, \tag{6.17b}$$

and
$$\bar{\zeta}_2^L = -\psi |A|^2 + \hat{w} \quad \text{on} \quad z = 0. \tag{6.17c}$$

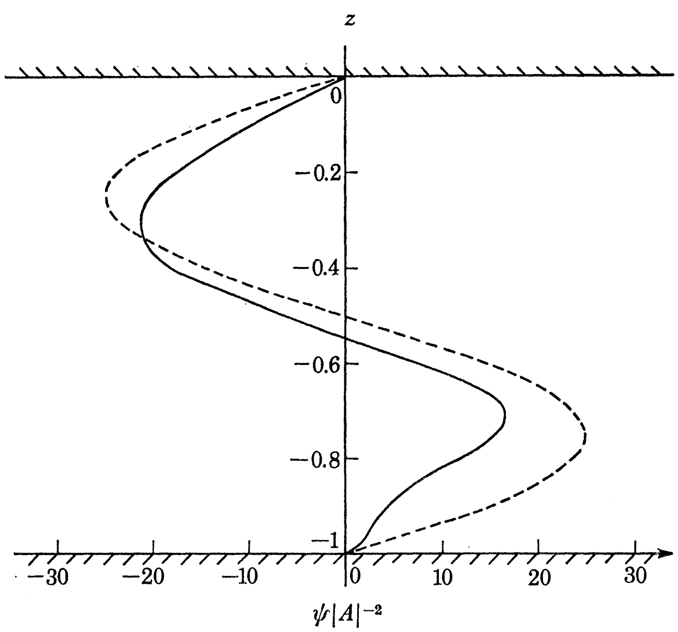

FIGURE 3. A plot of $\psi|A|^{-2}$ against z for the case $N = 1$, and $u_0 = U_1 z$, with the Boussinesq approximation. The depth $h = 1$ and the wavenumber $\kappa = 3.5$; -----, the case $U_1 = 0$, when $c = 0.21$ and $V = 0.095$; ——, the case $U_1 = -0.1$, when $c = 0.26$ and $V = 0.14$. In both cases the eigenfunction f (6.1) has been normalized so that max $|f| = 1$.

Equation (6.16c) determines $\hat{u}$ only to within an arbitrary function of X, Y; this arbitrary function can be determined by putting

$$Q = \int_{-h}^{0} \hat{u}\,\mathrm{d}z, \tag{6.18}$$

and obtaining the following equation for Q, by eliminating $\bar{p}_2$ and $\bar{v}_2^L$ from (6.4a, b, d),

$$\frac{\partial^2 Q}{\partial X^2}+\frac{\partial^2 Q}{\partial Y^2} = \frac{\partial^2}{\partial X^2}\{(u_0-V)\,\hat{w}\}_{z=0}+\frac{\partial^2}{\partial Y^2}\int_{-h}^{0}\hat{w}\,\frac{\partial u_0}{\partial z}\,\mathrm{d}z$$
$$+\frac{\partial^2|A|^2}{\partial Y^2}\int_{-h}^{0}\left\{2\omega^*\kappa\left(f^2+\frac{1}{\kappa^2}\left(\frac{\partial f}{\partial z}\right)^2\right)-(u_0-V)\,\frac{\partial^2 f^2}{\partial z^2}-(u_0-V)\,\frac{\partial\psi}{\partial z}\right\}\mathrm{d}z. \tag{6.19}$$

This is a Poisson equation for Q similar to an equation found by Davey & Stewartson (1974) in their study of the transverse modulations of a water wave packet. By using equations (4.9a, b) for the Stokes velocities, (6.16a) and (6.19) agree with the equations obtained by Grimshaw (1977) for the transverse modulations of an internal gravity wave packet in the absence of a shear flow.

To solve (6.15a) we shall use a Fourier transform in X, Y. Thus, we suppose that

$$\hat{w},\ \text{etc.} \propto \exp\,(\mathrm{i}LX+\mathrm{i}MY), \tag{6.20}$$

and then (6.16a) becomes

$$L^2\left\{\frac{\partial}{\partial z}\left(\rho_0(u_0-V)^2\,\frac{\partial\hat{w}}{\partial z}\right)+\rho_0 N^2\hat{w}\right\}+M^2\rho_0 N^2\hat{w} = \mathscr{M}_2 M^2|A|^2. \tag{6.21}$$

Comparing the homogeneous equation (6.21) (i.e. the right hand side is replaced by zero) with the eigenvalue equation (6.2a) in the limit $\kappa\to 0$, and also comparing the corresponding boundary conditions, we see that the free solutions for $\hat{w}$ will be long waves if (sec ϕ) $c_0(\phi) = V$, where $c_0(\phi)$ is the phase speed of a long wave inclined at an angle ϕ to the shear flow $u_0(z)$, and ϕ is the angle between the wave number (L, M) and the X-axis (note that $c_0(\phi)$ is the eigenvalue of the long wave eigenvalue problem obtained by replacing u_0 by $u_0\cos\phi$ in (6.1) and then choosing $\kappa_1 = \kappa$, $\kappa_2 = 0$ and taking the limit $\kappa\to 0$). When this occurs the inhomogeneous equation (6.21) cannot be solved for $\hat{w}$, and there is an oblique long wave resonance between the wave packet and a long wave mode. Equations describing this resonance have been developed by Grimshaw (1977) for the case $u_0 = v_0 = 0$; the corresponding equations in the present case will be described elsewhere. Otherwise (6.21) has solutions similar to those described for (6.8a), and there are again significantly large flows with a fine microstructure whenever $|u_0-V|$ is small in the flow domain.

References

Andrews, D. G. & McIntyre, M. E. 1978a An exact theory of nonlinear waves on a Lagrangian-mean flow. *J. Fluid Mech.* **89**, 609–646.

Andrews, D. G. & McIntyre, M. E. 1978b On wave-action and its relatives. *J. Fluid Mech.* **89**, 647–664.

Booker, J. R. & Bretherton, F. P. 1967 The critical layer for internal gravity waves in a shear flow. *J. Fluid Mech.* **27**, 513–539.

Bretherton, F. P. & Garrett, C. J. R. 1969 Wavetrains in inhomogeneous moving media. *Proc. R. Soc. Lond.* A **302**, 529–554.

Bretherton, F. P. 1971 The general linearised theory of wave propagation. *Lect. appl. Math.* **13**, 61–102.

Banks, W. H., Drazin, D. G. & Zaturska, M. B. 1976 On the normal modes of parallel flow of inviscid stratified fluid. *J. Fluid Mech.* **75**, 149–171.

Chimonas, G. 1978 Packet scale motions forced by non-linearities in a wave system. *J. atmos. Sci.* **35**, 382–396.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Davey, A. & Stewartson, K. 1974 On three-dimensional packets of surface waves. *Proc. R. Soc. Lond.* A **338**, 101–110.
Dewar, R. L. 1970 Interaction between hydromagnetic waves and a time-dependent inhomogeneous medium. *Phys. Fluids* **13**, 2710–2720.
Dore, B. D. 1970 On progressive internal waves in non-uniform depths. *Pure appl. Geophys.* **83**, 87–97.
Garrett, C. J. R. 1968 On the interaction between internal gravity waves and a shear flow. *J. Fluid Mech.* **34**, 711–720.
Grimshaw, R. 1977 The modulation of an internal gravity-wave packet, and the resonance with the mean motion. *Stud. appl. Math.* **56**, 241–266.
Grimshaw, R. 1978 Nonlinear wave packets, wave action and mean motions. *Univ. Melb. Math. Res. Rep.* **18.**
Hayes, W. D. 1970 Conservation of action and modal wave action. *Proc. R. Soc. Lond.* A **320**, 187–208.
Hector, D., Cohen, J. & Bleistein, N. 1972 Ray method expansions for surface and internal waves in inhomogeneous oceans of variable depth. *Stud. appl. Math.* **51**, 121–137.
Hogg, N. G. 1971 Longshore current generation by obliquely incident internal waves. *Geophys. Fluid Dynamics* **2**, 361–376.
Keller, J. B. & Mow, V. C. 1969 Internal wave propagation in a inhomogeneous fluid of non-uniform depth. *J. Fluid Mech.* **38**, 365–374.
Leonov, A. I., Miropolsky, Yu. Z., Voronovich, A. G. & Borisenko, Yu. D. 1979 Evolution of weak nonlinear internal waves. *J. Fluid Mech.* (in the press).
McIntyre, M. E. 1973 Mean motions and impulse of a guided internal gravity wave packet. *J. Fluid Mech.* **60**, 801–811.
McKee, W. D. 1973 Internal-inertia waves in a fluid of variable depth. *Proc. Camb. phil. Soc.* **73**, 205–213.
McKee, W. D. 1974 Waves on a shearing current: a uniformly valid asymptotic solution. *Proc. Camb. phil. Soc.* **75**, 295–301.
Thorpe, S. 1977 On the stability of internal wavetrains. In *Deep sea research, a voyage of discovery* (ed. M. V. Angel), pp. 199–212.
Thorpe, S. 1978 On the shape and breaking of finite amplitude internal gravity waves in a shear flow. *J. Fluid Mech.* **85**, 7–31.
Wunsch, C. 1969 Progressive internal waves on slopes. *J. Fluid Mech.* **35**, 131–144.
Wunsch, C. 1971 Note on some Reynolds stress effects of internal waves on slopes. *Deep Sea Res.* **18**, 583–591.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES